Radar Time and a State-Space Based Approach To Quantum Field Theory In Gravitational and Electromagnetic Backgrounds

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In a recent paper [1] a new initial value formulation of fermionic QFT was presented that is applicable to an arbitrary observer in any electromagnetic background. This approach suggests a consistent particle interpretation at all times, with the concept of 'radar time' used to generalise this interpretation to an arbitrarily moving observer. In the present paper we extend this formalism to allow for gravitational backgrounds. The observer-dependent particle interpretation generalises Gibbons' definition [2] to non-stationary spacetimes. This allows any observer to be considered, providing a particle interpretation that depends *only* on the observer's motion and the background, not on any choice of coordinates or gauge, or on details of their particle detector. Consistency with known results is demonstrated for the cases of Rindler space and deSitter space. Radar time is also considered for an arbitrarily moving observer in an arbitrary 1+1 dimensional spacetime, and for a comoving observer in a 3+1 dimensional FRW universe with arbitrary scale factor $a(t)$. Finite volume measurements and their fluctuations are also discussed, allowing one to say with definable precision where and when the particles are observed.

Key Words: particle creation, fermion, observer, Slater determinant, radar time.

1. INTRODUCTION

Our recent initial value formulation of fermionic QFT [1] emphasised the states of the system, described in terms of Slater determinants of Dirac states. The vacuum was defined as the Slater determinant of a basis for the span of the negative spectrum of the 'first quantized' Hamiltonian, thus providing a concrete manifestation of the Dirac Sea. As well as generating simple derivations of the general S-Matrix element and expectation value in the theory, the approach suggested a consistent particle interpretation at all times. In the present paper we extend this work to encompass gravitational backgrounds.

In common with the Canonical, Tunnelling, and Wave-functional approaches, our analysis depends explicitly on the choice of 'in' and 'out' particle/antiparticle decomposition, and on the corresponding categorisation of in and out modes. Various categorisation schemes have been proposed, based on asymptotic or adiabatic properties of solutions, or on the diagonalisation of a suitable Hamiltonian. However, most schemes depend on a choice of coordinates or of gauge [3], and the relation of these choices to the motion of an observer or the behaviour of a particle detector is often ambiguous [4]. Also, those schemes based on asymptotic or adiabatic approximations generate accurate predictions only at asymptotically late times or in sufficiently weak backgrounds. Particle detectors provide a more operational particle concept, but their predictions are not always proportional to the number of particles present [4, 5] even when the detector is inertial.

The categorisation scheme proposed in [1] and developed here provides a consistent particle interpretation at all times without requiring any asymptotic conditions on the in and out states. It consistently combines the conventional 'Bogoliubov coefficient' and 'tunnelling amplitude' methods, resolving gauge inconsistencies that trouble each of these [3]. Also, by utilising the concept of radar time, the present definition naturally incorporates the motion of the observer (detector), providing a definition which depends only on the observer's motion and on the background, and not on the choice of coordinates or gauge. Since it is applicable at all times, we can state not only how many particles are created, but also when they were created, and how they behaved after their creation (as in the application of [1] to spatially uniform electric fields [6]). By considering finite-volume operators with controllable fluctuations, we can further specify, with definable precision, where the particles are created.

In Section 2 we specify the representation of states used, and their evolution, and present formulae for the general S-Matrix element and the expectation value of the theory. The particle interpretation is described in Section 3, specifying the states of Section 2 in terms of their particle content. Finite volume measurements are also discussed in this Section, along with their fluctuations. For concreteness we suppose that the spacetime is globally hyperbolic, and that the observer-dependent foliation is Cauchy. Strictly, this excludes situations with particle horizons, but we present a simple example in Section 4 which shows that the formalism is still well suited to the treatment of horizons. We consider massive and massless Dirac fermions in 1+1 dimensional flat space as seen by a uniformly accelarating observer, and demonstrate consistency with known techniques, by rederiving the well-known thermal distribution of Rindler particles. We consider the spatial distribution of these Rindler particles, and comment briefly on fluctuations.

In Section 5 we present examples of radar time, treating an arbitrary observer in an arbitrary 1+1 dimensional spacetime, and also a comoving observer in an FRW universe with arbitrary scale factor $a(t)$. DeSitter space and the Milne universe are described in more detail. We conclude with a brief discussion in Section 6. An appendix describes the connection between projection operators and 2-point functions, and emphasises the role of the negative energy Wightman function as the "Dirac density matrix of the Dirac Sea".

2. THE STATE SPACE

2.1. Preliminaries

The Lagrangian density for the Dirac equation in gravitational and electromagnetic backgrounds is [7, 8]

$$\mathcal{L} = \Re[e(x)\bar{\psi}(x)(i\gamma^\mu(x)\nabla_\mu - m)\psi(x)] \tag{1}$$

where $\gamma^\mu(x) = e^\mu_a(x)\bar{\gamma}^a$ satisfies $\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x)$, $\bar{\gamma}^a$ are a representation of the normal flat space Dirac matrices, $\bar{\psi} \equiv \psi^\dagger\bar{\gamma}^0$, $e^\mu_a(x)$ is the *Vierbein*, and $e(x) \equiv \det(e^a_\mu(x)) = \sqrt{-\det(g_{\mu\nu}(x))}$. The covariant derivative ∇_μ acts on spinors according to:

$$\nabla_\mu\psi(x) = (\partial_\mu + \Gamma_\mu + ieA_\mu)\psi(x)$$

where $\Gamma_\mu = \frac{1}{4}\gamma_\nu\nabla_\mu\gamma^\nu$, A_μ is the electromagnetic potential, and e is the charge of the fermion ($e < 0$ for electrons). Γ_μ is often written [9] in terms of the 'connection field' $\omega^{ab}_\mu(x)$ as $\Gamma_\mu = -\frac{i}{4}\omega^{ab}_\mu\sigma_{ab}$ where $\sigma_{ab} = \frac{i}{2}[\bar{\gamma}_a, \bar{\gamma}_b]$.

This Lagrangian gives rise to the governing equation:

$$(i\gamma^\mu(x)\nabla_\mu - m)\psi(x) = 0 \tag{2}$$

and the energy-momentum tensor:

$$T_{\mu\nu}(\psi) = \Re[i\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi] - \frac{g_{\mu\nu}}{e(x)}\mathcal{L} \tag{3}$$

The inner product $\langle\psi|\phi\rangle_\Sigma$ on the spacelike Cauchy surface Σ is given by:

$$\langle\psi|\phi\rangle_\Sigma = \int e(x)\bar{\psi}(x)\gamma^\mu(x)\phi(x)d\Sigma_\mu \tag{4}$$

and is independent of Σ by virtue of (2).

The 'first quantized' state space $\mathcal{H}(\Sigma)$ on some spacelike hypersurface Σ can be defined as the space of all (finite norm) spinor valued functions of $x|_\Sigma$ (the projection of x^μ onto the hypersurface Σ). We restrict our attention to hypersurfaces Σ that are Cauchy, so that the various $\mathcal{H}(\Sigma)$ are all unitarily equivalent, and can simply be denoted $\mathcal{H}$. We will denote a first quantized state on Σ by $\psi(x|_\Sigma)$ or $|\psi_\Sigma\rangle$ or, where no ambiguity is possible, simply ψ. (There will be little need to distinguish between a state and its coordinate representation.)

2.2. The Full Fock Space Over $\mathcal{H}$

The *antisymmetric Fock Hilbert space* over the complex Hilbert space $\mathcal{H}$ is denoted $\mathcal{F}_\wedge(\mathcal{H})$ and is defined [10] in terms of the *antisymmetric Tensor Algebra* over $\mathcal{H}$. It is a natural and familiar construction by which a quantum theory of fermions can be formulated. This construction is described in [1], and as a preliminary we outline it here. Let $\mathcal{H}$ be the Hilbert space in the previous Section, with inner product denoted by $\langle\ |\ \rangle$. Let $\otimes^n\mathcal{H}$ denote the direct product of n copies of $\mathcal{H}$, and let $\wedge^n\mathcal{H}$ denote the restriction of $\otimes^n\mathcal{H}$ to those states which are completely antisymmetric under changes in the order of the elements $|\psi\rangle\in\mathcal{H}$ from which the state is constructed. Given $|\psi_1\rangle|\psi_2\rangle\ldots|\psi_n\rangle\in\otimes^n\mathcal{H}$, we can define $|\psi_1\wedge\psi_2\wedge\cdots\wedge\psi_n\rangle\in\wedge^n\mathcal{H}$ by:

$$|\psi_1\wedge\psi_2\wedge\cdots\wedge\psi_n\rangle\equiv\frac{1}{\sqrt{n!}}\sum_\sigma \mathrm{sign}(\sigma)|\psi_{\sigma(1)}\rangle|\psi_{\sigma(2)}\rangle\ldots|\psi_{\sigma(n)}\rangle \tag{5}$$

where $\{\sigma(i), i=1,\ldots,n\}$ is a permutation of $\{1\ldots n\}$. This is simply the Slater determinant of the states $|\psi_1\rangle\ldots|\psi_n\rangle$. The antisymmetric Fock Hilbert space $\mathcal{F}_\wedge(\mathcal{H})$ is now given by:

$$\mathcal{F}_\wedge(\mathcal{H})=\oplus_{n=0}^{\infty}\wedge^n\mathcal{H}$$

where $\wedge^0\mathcal{H}\equiv\mathbb{C}$ and $\wedge^1\mathcal{H}\equiv\mathcal{H}$. States which lie entirely within $\wedge^r\mathcal{H}$ for some r are said to be of *grade* r.

A useful operation on $\mathcal{F}_\wedge(\mathcal{H})$ is the 'inner derivative' $i_\psi:\wedge^n\mathcal{H}\to\wedge^{n-1}\mathcal{H}$ (named by analogy with differential geometry). This is defined by:

$$i_\psi|\phi_1\wedge\cdots\wedge\phi_n\rangle\equiv\sum_i(-)^{i+1}\langle\psi|\phi_i\rangle|\phi_1\wedge\cdots\wedge\check{\phi}_i\wedge\cdots\wedge\phi_n\rangle \tag{6}$$

where the check over ϕ_i signifies that this state is omitted from the product. The relation $i_\psi:\mathcal{F}_\wedge(\mathcal{H})\to\mathcal{F}_\wedge(\mathcal{H})$ is obtained from (6) by imposing linearity, together with the convention $i_\psi\lambda=0$ for $\lambda\in\wedge^0\mathcal{H}$. It is clear that $i_\psi(i_\psi|F\rangle)=0$ for all $|F\rangle\in\mathcal{F}_\wedge(\mathcal{H})$, and that:

$$i_\psi(\phi\wedge|F\rangle)=\langle\psi|\phi\rangle|F\rangle-\phi\wedge(i_\psi|F\rangle) \tag{7}$$

The operation i_ψ is denoted $a(\psi)$ by Ottesen [10], and plays the role of an annihilation operator. Here i_ψ will play a similar, although not identical role.

Finally, the inner product on $\mathcal{F}_\wedge(\mathcal{H})$ is given by:

$$\langle \psi_1 \wedge \cdots \wedge \psi_n | \phi_1 \wedge \cdots \wedge \phi_m \rangle = \delta_{nm} \det[\langle \psi_i | \phi_j \rangle] \tag{8}$$

where $\langle \psi_i | \phi_j \rangle$ refers to the inner product on $\mathcal{H}$. (For states $\lambda, \mu \in \wedge^0 \mathcal{H}$ define $\langle \lambda | \mu \rangle = \bar{\lambda}\mu$ and $\langle \lambda | F_n \rangle = 0$ for any state $|F_n\rangle$ of grade $n > 0$.) This definition agrees with the inner product defined in terms of Slater determinants. Although we use the notation $\langle \; | \; \rangle$ to refer to both the inner product on $\mathcal{H}$ and the inner product on $\mathcal{F}_\wedge(\mathcal{H})$, it will always be clear from the context which is involved.

2.3. Operators on Fock Space

Let $\hat{A}_1 : \mathcal{H} \to \mathcal{H}$ be an operator on the space of Dirac states. We wish to construct from it an operator which can act on all of state space. There are two useful ways of doing this: *Hermitian extension* $\hat{A}_H : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$, and *Unitary extension* $\hat{A}_U : \mathcal{F}_\wedge(\mathcal{H}) \to \mathcal{F}_\wedge(\mathcal{H})$ (outlined also in Ottesen [10]). These are respectively defined by:

$$\hat{A}_H |\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n\rangle \equiv \sum_{i=1}^{n} |\psi_1 \wedge \ldots (\hat{A}_1 \psi_i) \wedge \psi_{i+1} \cdots \wedge \psi_n\rangle \tag{9}$$

$$\hat{A}_U |\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n\rangle \equiv |(\hat{A}_1 \psi_1) \wedge (\hat{A}_1 \psi_2) \wedge \cdots \wedge (\hat{A}_1 \psi_n)\rangle \tag{10}$$

If $\hat{A}_1$ is (anti)hermitian with respect to the inner product (4) on $\mathcal{H}$, then $\hat{A}_H$ is (anti)hermitian with respect to the inner product (8) on $\mathcal{F}_\wedge(\mathcal{H})$. If $\hat{A}_1$ is unitary, then so is $\hat{A}_U$. Also $(e^{\hat{A}_1})_U = e^{\hat{A}_H}$, so that if $\hat{U}_1 = e^{\hat{A}_1}$ on $\mathcal{H}$ then $\hat{U}_U = e^{\hat{A}_H}$ on $\mathcal{F}_\wedge(\mathcal{H})$.

Some Simple Properties

1. $(\hat{A} + \hat{B})_H = \hat{A}_H + \hat{B}_H$, $[\hat{A}_H, \hat{B}_H] = [\hat{A}, \hat{B}]_H$ and $(\hat{A}\hat{B})_U = \hat{A}_U \hat{B}_U$.

2. $[\hat{A}_H, \psi \wedge] = (\hat{A}_1 \psi)$ and $[\hat{A}_H, i_\psi] = -i_{\hat{A}_1^\dagger \psi}$

3. If $\psi_1, \psi_2, \ldots \psi_n$ are all eigenstates of $\hat{A}_1$ with eigenvalues $\lambda_1, \ldots \lambda_n$, then $|\psi_1 \wedge \psi_2 \cdots \wedge \psi_n\rangle$ is an eigenstate of $\hat{A}_H$ with eigenvalue $\sum_{i=1}^n \lambda_i$.

4. If $\psi_1, \psi_2, \dots \psi_n$ are orthonormal and $|F\rangle \equiv |\psi_1 \wedge \psi_2 \cdots \wedge \psi_n\rangle$ then

$$\langle F|\hat{A}_H|F\rangle = \sum_{i=1}^{n} \langle\psi_i|\hat{A}_1|\psi_i\rangle \tag{11}$$

$$\langle F|(\hat{A}_H)^2|F\rangle = \sum_{i=1}^{n} \langle\psi_i|\hat{A}_1^2|\psi_i\rangle + 2\sum_{i<j} (\langle\psi_i|\hat{A}_1|\psi_i\rangle\langle\psi_j|\hat{A}_1|\psi_j\rangle - \langle\psi_i|\hat{A}_1|\psi_j\rangle\langle\psi_j|\hat{A}_1|\psi_i\rangle) \tag{12}$$

$$\langle F|(\hat{A}_H)^2|F\rangle - (\langle F|\hat{A}_H|F\rangle)^2 = \sum_{i} \langle\psi_i|\hat{A}_1^2|\psi_i\rangle - \sum_{i,j} |\langle\psi_i|\hat{A}_1|\psi_j\rangle|^2 \tag{13}$$

2.4. Evolution of States

Given two Cauchy surfaces Σ_1 and Σ_0, we can define the evolution operator $\hat{U}_1(\Sigma_1,\Sigma_0)$ on $\mathcal{H}$ by

$$\hat{U}_1(\Sigma_1,\Sigma_0)|\psi_{\Sigma_0}\rangle \equiv |\psi_{\Sigma_0}(\Sigma_1)\rangle \tag{14}$$

where $|\psi_{\Sigma_0}\rangle$ represents some chosen initial conditions $\psi(x|_{\Sigma_0})$ on Σ_0, and $|\psi_{\Sigma_0}(\Sigma_1)\rangle$ represents the corresponding solution $\psi(x)$ of the Dirac equation, expressed on the hypersurface Σ_1.

We consider only QFT in an (external) gravitational or electromagnetic background, so that we ignore direct particle-particle interactions and work within the 'zeroth order Hartree-Fock' approximation. This assumes that the evolution operator on $\mathcal{F}_\wedge(\mathcal{H})$ is just the unitary extension of the evolution operator on $\mathcal{H}$. The action of $\hat{U}(\Sigma_1,\Sigma_0)$ is now given by:

$$\hat{U}(\Sigma_1,\Sigma_0)|\psi_{1,\Sigma_0} \wedge \cdots \wedge \psi_{n,\Sigma_0}\rangle = |\psi_{1,\Sigma_0}(\Sigma_1) \wedge \cdots \wedge \psi_{n,\Sigma_0}(\Sigma_1)\rangle \tag{15}$$

The multiparticle solution is simply the Slater determinant of the appropriate 'first quantized' solutions. This construction preserves grade, and implies that the unitarity of $\hat{U}(\Sigma_1,\Sigma_0)$ follows immediately from the unitarity of the first quantized Dirac equation.

We have now set up a state space, an evolution equation (15) and a conserved inner product (8). These are all we need to calculate arbitrary S-Matrix elements (from (15) and (8)), arbitrary expectation values (from (11)), and even fluctuations in these expectation values (from (13)). However, the theory is not invested with physical meaning until the states of the

system can be specified in terms of their physical properties. For this purpose a particle interpretation is invaluable. We now introduce an observer-dependent particle interpretation. This will rely on the observer-dependent foliation of spacetime provided by Bondi's Radar Time [11, 12, 6].

3. OBSERVER DEPENDENT PARTICLE INTERPRETATION

3.1. Bondi's Radar Time

Consider an observer travelling on path $\gamma : x^\mu = x^\mu(\tau)$ with proper time τ, and define:

$\tau^+(x) \equiv$ (earliest possible) proper time at which a null geodesic leaving point x could intercept γ.

$\tau^-(x) \equiv$ (latest possible) proper time at which a null geodesic could leave γ, and still reach point x.

$\tau(x) \equiv \frac{1}{2}(\tau^+(x) + \tau^-(x)) \quad$ = 'radar time'.

$\rho(x) \equiv \frac{1}{2}(\tau^+(x) - \tau^-(x)) \quad$ = 'radar distance'.

$\Sigma_{\tau_0} \equiv \{x : \tau(x) = \tau_0\} =$ observer's 'hypersurface of simultaneity at time τ_0'.

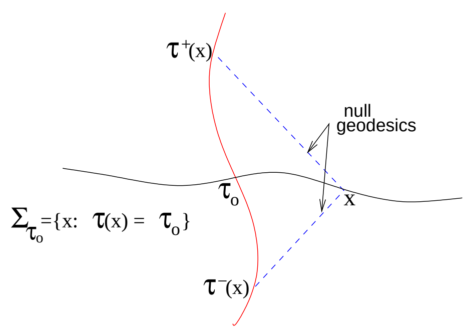

FIG. 1. Schematic of the definition of 'radar time' $\tau(x)$.

This is a simple generalisation of the definition made popular by Bondi in his work on special relativity and k-calculus [11, 12, 13]. By construction radar time is independent of the choice of coordinates (since no coordinates need be introduced in defining it) and it depends only on the motion of the observer. It agrees with proper time on the observer's path, and is invariant under 'time-reversal' - that is, under reversal of the sign of the observer's proper time.

It is clear from the definition of radar time that Σ_{τ_1} lies to the future of Σ_{τ_0} (for $\tau_1 > \tau_0$) except at the observer's particle horizon (if one exists), on which the various Σ_τ converge. When horizons are present, the domain of $\tau(x)$ is no longer all of spacetime - only that part with which the observer can send and receive signals.

Define now the 'time-translation' vector field:

$$k_\mu(x) \equiv \frac{\frac{\partial\tau}{\partial x^\mu}}{g^{\sigma\nu}\frac{\partial\tau}{\partial x^\sigma}\frac{\partial\tau}{\partial x^\nu}} \tag{16}$$

This represents the perpendicular distance between neighbouring hypersurfaces of simultaneity, since it is normal to these hypersurfaces and it satisfies $k^\mu(x)\frac{\partial\tau}{\partial x^\mu} = 1$. Now use the identity $i\not{k}\gamma^\mu\nabla_\mu = ik^\mu\nabla_\mu + \sigma^{\mu\nu}k_\mu\nabla_\nu$ (where $\not{k} \equiv k_\mu\gamma^\mu$ and $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \sigma^{ab}e_a^\mu e_b^\nu$) to write the Dirac equation as:

$$ik^\mu\nabla_\mu\psi = \hat{H}_{\mathrm{nh}}(\tau)\psi \equiv -\sigma^{\mu\nu}k_\mu\nabla_\nu\psi + m\not{k}\psi \tag{17}$$

Here $\hat{H}_{\mathrm{nh}}(\tau)$ is not in general Hermitian! (hence the subscript nh). At first sight this seems to disagree with unitarity on $\mathcal{H}$. However there is no inconsistency, because the inner product now depends explicitly on τ (via the volume element on Σ_τ), and since (17) is no longer of the form $i\frac{d}{dt}|\psi(t)\rangle = \hat{H}_1(t)|\psi(t)\rangle$, invalidating the standard equivalence proof of unitary evolution and a Hermitian Hamiltonian.

To investigate the relation between $\hat{H}_{\mathrm{nh}}(\tau_0)$ and the energy momentum tensor, define:

$$H_{\tau_0}(\psi) \equiv \int_{\Sigma_{\tau_0}} T_{\mu\nu}(\psi(x))k^\mu \mathrm{d}\Sigma^\nu \tag{18}$$

Substitution of (3) into this gives:

$$H_{\tau_0}(\psi) = \Re[\langle\psi|\hat{H}_{\mathrm{nh}}(\tau_0)|\psi\rangle_{\Sigma_{\tau_0}}] = \langle\psi|\hat{H}_1(\tau_0)|\psi\rangle_{\Sigma_{\tau_0}} \tag{19}$$
$$\text{where } \hat{H}_1(\tau_0) \equiv \tfrac{1}{2}\{\hat{H}_{\mathrm{nh}}(\tau_0) + \hat{H}^\dagger_{\mathrm{nh}}(\tau_0)\}$$

We can define the projection operators $\hat{P}^\pm_{\tau_0} : \mathcal{H} \to \mathcal{H}^\pm(\tau_0)$, and hence the spaces $\mathcal{H}^\pm(\tau_0)$, by requiring that $\hat{P}^\pm_{\tau_0}$ are orthogonal projections satisfying:

$$H_{\tau_0}(\hat{P}^+_{\tau_0}\psi) \geq H_{\tau_0}(\psi) \geq H_{\tau_0}(\hat{P}^-_{\tau_0}\psi) \tag{20}$$

for all $\psi \in \mathcal{H}$, as in [14]. This definition depends only on the background and the motion of the observer, and not on the choice of coordinates or gauge. It is equivalent to defining:

$$\mathcal{H}^+(\tau_0) \text{ as the span of the positive spectrum of } \hat{H}_1(\tau_0)$$
$$\mathcal{H}^-(\tau_0) \text{ as the span of the negative spectrum of } \hat{H}_1(\tau_0)$$

Here $\mathcal{H}^+(\tau_0)$ is the set of all positive energy states, and $\mathcal{H}^-(\tau_0)$ is the set of all negative energy states as defined on Σ_{τ_0}. This definition generalises Gibbons' approach [2] to arbitrary observers and non-stationary spacetimes.

In the Canonical approach to background QFT, this split of $\mathcal{H}$ into positive/negative energy modes can be achieved by Hamiltonian diagonalisation of the second quantized Hamiltonian that arises by substituting the field operator $\hat{\psi}(x)$ into expression (18) for $H_{\tau_0}(\psi)$. Hamiltonian diagonalisation has been criticised [15] for its reliance on the choice of a second quantized Hamiltonian which depends on an apparently arbitrary choice of hypersurface and time-translation vector field k^μ. Here this arbitrariness has been resolved (and in [1, 16]) by specifying the hypersurface Σ_{τ_0} and the vector field k^μ in terms of the worldline of the observer (or particle detector).

We have assumed here that $\hat{H}_1(\tau)$ has no zero energy eigenstates (for any τ). Even for inertial observers, there exist topologically non-trivial backgrounds for which zero energy eigenstates exist, leading to the existence of fractional charge; see e.g. [17]. Although such situations are straightforward to describe within the present approach, we will not discuss them further here. When there is a particle horizon, so that the observer's hypersurfaces are not Cauchy, zero-energy eigenstates are plentiful and correspond to states that are unobservable by that observer. For concreteness we will suppose in this Section that all Σ_τ are Cauchy. In Section 4 we demonstrate with a simple example that this approach remains useful in the presence of horizons.

3.2. Particle States and S-Matrix Elements

Having defined $\mathcal{H}^\pm(\tau_0)$ for any given observer, we can now define their *vacuum* $|\text{vac}_{\tau_0}\rangle$ *on* Σ_{τ_0}, to be the Slater determinant of any basis of $\mathcal{H}^-(\tau_0)$ (normalised so that $\langle\text{vac}_{\tau_0}|\text{vac}_{\tau_0}\rangle = 1$). This specifies $|\text{vac}_{\tau_0}\rangle$ up to an arbitrary phase factor. It is the state in which all negative energy degrees of freedom are full, and hence is a concrete manifestation of the Dirac Sea.

To illustrate this, let $\{u_{i,\tau_0}; i \in I\}$, $\{v_{i,\tau_0}; i \in I\}$ be orthonormal bases for $\mathcal{H}^+(\tau_0)$ and $\mathcal{H}^-(\tau_0)$ respectively, where I is some countable index set (the

uncountable case introduces no complications). The vacuum on Σ_{τ_0} can be written as:

$$|\mathrm{vac}_{\tau_0}\rangle = |v_{1,\tau_0} \wedge v_{2,\tau_0} \wedge \ldots\rangle \tag{21}$$

and is independent of the choice of basis for $\mathcal{H}^-(\tau_0)$ (up to a phase factor) because of the complete antisymmetry of the Slater determinant. The vacuum on Σ_{τ_1} at some 'time' τ_1 can similarly be written as:

$$|\mathrm{vac}_{t_1}\rangle = |v_{1,\tau_1} \wedge v_{2,\tau_1} \wedge \ldots\rangle$$

The observer who prepares the state 'at time τ_0', on his hypersurface of simultaneity Σ_{τ_0}, needn't be the same observer who measures that state 'at time τ_1' (on her hypersurface Σ_{τ_1}). A typical case in which these two observers differ is the Unruh effect, where $|\mathrm{vac}_{\tau_0}\rangle$ is the Minkowski vacuum, prepared by an inertial observer, and $|\mathrm{vac}_{\tau_1}\rangle$ is the Rindler vacuum, defined in the frame of a uniformly accelerating observer.

The evolved state $|\mathrm{vac}_{\tau_0}(\tau_1)\rangle$ obtained by evolving $|\mathrm{vac}_{\tau_0}\rangle$ from Σ_{τ_0} to Σ_{τ_1} is (since $\hat{U} = \hat{U}_{1,U}$):

$$|\mathrm{vac}_{\tau_0}(\tau_1)\rangle = |v_{1,\tau_0}(\tau_1) \wedge v_{2,\tau_0}(\tau_1) \wedge \ldots\rangle \tag{22}$$

where $v_{i,\tau_0}(\tau_1)$ denotes the state (called $|\psi_{\Sigma_{\tau_0}}(\Sigma_{\tau_1})\rangle$ in (14)) obtained from v_{i,τ_0} by evolution to Σ_{τ_1}. It will not in general be contained in $\mathcal{H}^-(\tau_1)$. We will often refer to $|\mathrm{vac}_{\tau_0}(\tau_1)\rangle$ as the 'evolved vacuum', although it is not in general a vacuum state.

From (8), the vacuum-vacuum S-matrix element is simply:

$$\langle \mathrm{vac}_{\tau_1}|\mathrm{vac}_{\tau_0}(\tau_1)\rangle = \det[\langle v_{i,\tau_1}|v_{j,\tau_0}(\tau_1)\rangle] \tag{23}$$

The probability that $|\mathrm{vac}_{\tau_0}(\tau_1)\rangle$ will be vacuum at time τ_1 is then $\mathcal{P}_{\mathrm{vac}\to\mathrm{vac}} = |\langle \mathrm{vac}_{\tau_1}|\mathrm{vac}_{\tau_0}(\tau_1)\rangle|^2$. Although this result can be derived by a number of methods once a particle interpretation is specified [18, 19, 20], we believe that the present derivation (and in [1, 16]) is clearer and more economical.

States containing particles can be treated just as easily. For instance, a one-electron state (at time τ_0) is of the form $u_{\tau_0} \wedge |\mathrm{vac}_{\tau_0}\rangle$, and a one-positron state (at time τ_0) is of the form $i_{v_{\tau_0}}|\mathrm{vac}_{\tau_0}\rangle$. As expected, electrons are represented by the presence of positive energy degrees of freedom, and

positrons by the absence of negative energy degrees of freedom (note that $v_{\tau_0}\wedge|\mathrm{vac}_{\tau_0}\rangle = 0 = i_{u_{\tau_0}}|\mathrm{vac}_{\tau_0}\rangle$ for all $u_{\tau_0}\in\mathcal{H}^+(\tau_0)$ and $v_{\tau_0}\in\mathcal{H}^-(\tau_0)$). We introduce the symbol $|\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}_{\tau_0}\rangle$ to denote an 'in' state of m particles (in states $u_{i_1}\dots u_{i_m}$ with $i_1 < i_2 < \dots i_m$ by convention), and n antiparticles (corresponding to the absence of states $v_{j_1}\dots v_{j_n}$), prepared at time τ_0. This state is given by:

$$|\binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}_{\tau_0}\rangle \equiv (-)^J |u_{i_1,\tau_0}\wedge \dots u_{i_m,\tau_0}\wedge v_{1,\tau_0}\wedge \dots \check{v}_{j_1,\tau_0}\cdots\wedge \check{v}_{j_n,\tau_0}\dots\rangle \tag{24}$$

where the check over v_{j,τ_0} signifies that this degree of freedom is missing from the state, and $J = \frac{n}{2}(n+1) + \sum_{k=1}^n j_k$ appear as an unimportant sign convention.

The general S-matrix element can immediately be written as:

$$\begin{aligned}
&\langle \binom{i'_1 i'_2 \dots i'_{m'}}{j'_1 j'_2 \cdots j'_{n'}}_{\tau_1} | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}_{\tau_0}(\tau_1)\rangle \\
&= (-)^{J-J'} \det \begin{bmatrix}
\begin{bmatrix} \alpha_{i'_1 i_1} & \cdots & \alpha_{i'_1 i_m} \\ \vdots & & \vdots \\ \alpha_{i'_{m'} i_1} & \cdots & \alpha_{i'_{m'} i_m} \end{bmatrix} &
\begin{bmatrix} \beta_{i'_1 1} & \cdots & \binom{j_1 \dots j_n}{missing} \\ \vdots & & \vdots \\ \beta_{i'_{m'} 1} & \cdots & \binom{j_1 \dots j_n}{missing} \end{bmatrix} \\
\begin{bmatrix} \gamma_{1 i_1} & \cdots & \gamma_{1 i_m} \\ \vdots & & \vdots \\ \binom{j'_1 \dots j'_{n'}}{missing} & \cdots & \binom{j'_1 \dots j'_{n'}}{missing} \end{bmatrix} &
\begin{bmatrix} \epsilon_{11} & \cdots & \binom{j_1 \cdots j_n}{missing} \\ \vdots & & \\ \binom{j'_1 \dots j'_{n'}}{missing} & & \end{bmatrix}
\end{bmatrix}
\end{aligned} \tag{25}$$

(if $m - n = m' - n'$, and zero otherwise), where

$$\alpha_{ij}(\tau_1,\tau_0) = \langle u_{i,\tau_1}|u_{j,\tau_0}(\tau_1)\rangle \qquad \gamma_{ij}(\tau_1,\tau_0) = \langle v_{i,\tau_1}|u_{j,\tau_0}(\tau_1)\rangle \tag{26}$$

$$\beta_{ij}(\tau_1,\tau_0) = \langle u_{i,\tau_1}|v_{j,\tau_0}(\tau_1)\rangle \qquad \epsilon_{ij}(\tau_1,\tau_0) = \langle v_{i,\tau_1}|v_{j,\tau_0}(\tau_1)\rangle \tag{27}$$

are the *time-dependent Bogoliubov coefficients*. The Bogoliubov conditions follow from unitarity of the 'first quantized' evolution matrix

$$\boldsymbol{S}_1(\tau_1,\tau_0) = \begin{bmatrix} \boldsymbol{\alpha}(\tau_1,\tau_0) & \boldsymbol{\beta}(\tau_1,\tau_0) \\ \boldsymbol{\gamma}(\tau_1,\tau_0) & \boldsymbol{\epsilon}(\tau_1,\tau_0) \end{bmatrix}$$

The ease with which the general S-Matrix element (25) has been derived contrasts with many conventional formulations [21, 22, 20, 23], as discussed in detail in [1]. The reason is in the concrete representation of states given by equations such as (21) and (24), and the simple evolution equation which allows us to deduce equations such as (22). In Canonical approaches to

QFT in a classical background [7, 33, 23] the states are defined implicitly, by the requirement $a_i|\text{vac}\rangle = 0 = \mathrm{b_i}|\text{vac}\rangle$, where the creation/annihilation operators a_i,b_i are defined implicitly by the CAR's. The derivation of S-Matrix elements then involves more round about methods. One such method, analogous to that used in [23], is described in [1] and contrasted with the derivation above.

3.3. Expectation Values

Given an operator $\hat{A}_1(\tau):\mathcal{H}\to\mathcal{H}$, we can define its *physical extension* $\hat{A}_{\text{phys}}(\tau):\mathcal{F}_\wedge(\mathcal{H})\to\mathcal{F}_\wedge(\mathcal{H})$ by:

$$\hat{A}_{\text{phys}}(\tau) = \hat{A}_H(\tau) - \langle\text{vac}_\tau|\hat{A}_H(\tau)|\text{vac}_\tau\rangle\hat{1} \tag{28}$$

This is the relativistic equivalent of the 'one-particle operator' of multiparticle quantum mechanics [30], and is expressible as a normal-ordered bilinear of the field operator $\hat{\psi}(x)$. This vacuum subtraction is equivalent to normal ordering with respect to the particle interpretation *at the time of measurement*. This choice is also made in previous 'Hamiltonian diagonalisation' procedures [24, 25, 26, 27, 28], and uniquely guarantees the positive definiteness of $\hat{H}_{\text{phys}}(\tau)$ while maintaining $\langle\text{vac}_\tau|\hat{H}_{\text{phys}}(\tau)|\text{vac}_\tau\rangle = 0$. This is discussed in more detail in [1], where the relation of our approach to Hamiltonian Diagonalisation is also described.

The expectation value of $\hat{A}_{\text{phys}}(\tau)$ in the physical vacuum $|\text{vac}_\tau\rangle$ at time τ is zero by construction. Its expectation value in the 'evolved vacuum' $|\text{vac}_{\tau_0}(\tau)\rangle$ is in general non-zero, and takes the form

$$\begin{aligned}
&\langle\text{vac}_{\tau_0}(\tau_1)|\hat{A}_{\text{phys}}(\tau_1)|\text{vac}_{\tau_0}(\tau_1)\rangle && (29)\\
&\quad= \sum_{i=1}^{N}\langle v_{i,\tau_0}(\tau_1)|\hat{A}_1(\tau_1)|v_{i,\tau_0}(\tau_1)\rangle - \sum_{i=1}^{N}\langle v_{i,\tau_1}|\hat{A}_1(\tau_1)|v_{i,\tau_1}\rangle && (30)\\
&\quad= \text{Trace}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger\boldsymbol{A}^{++} - \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger\boldsymbol{A}^{--} + \boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger\boldsymbol{A}^{+-} + \boldsymbol{\beta}\boldsymbol{\epsilon}^\dagger\boldsymbol{A}^{-+}) && (31)
\end{aligned}$$

where we have defined:

$$\begin{aligned}
\boldsymbol{A}^{++}_{jk} &\equiv \langle u_{j,\tau_1}|\hat{A}_1(\tau_1)|u_{k,\tau_1}\rangle \qquad & \boldsymbol{A}^{--}_{jk} &\equiv \langle v_{j,\tau_1}|\hat{A}_1(\tau_1)|v_{k,\tau_1}\rangle\\
\boldsymbol{A}^{+-}_{jk} &\equiv \langle u_{j,\tau_1}|\hat{A}_1(\tau_1)|v_{k,\tau_1}\rangle \quad \text{and} & \boldsymbol{A}^{-+}_{jk} &\equiv \langle v_{j,\tau_1}|\hat{A}_1(\tau_1)|u_{k,\tau_1}\rangle\\
& & &= \overline{\boldsymbol{A}^{+-}_{kj}} \text{ if } \hat{A}_1 \text{ is Hermitian}
\end{aligned} \tag{32}$$

The relation $(\boldsymbol{\epsilon}\boldsymbol{\epsilon}^\dagger)_{kj} - \delta_{kj} = -(\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{kj}$ has been used in deriving (31) from (30). This step relies on the fact that we are vacuum subtracting

with respect to the vacuum *at the time of measurement.* Notice that if $\hat{A}_1$ is conserved at the level of the Dirac equation, then:

$$\langle \mathrm{vac}_{\tau_0}(\tau_1)|\hat{A}_{\mathrm{phys}}|\mathrm{vac}_{\tau_0}(\tau_1)\rangle = \mathrm{Trace}(\boldsymbol{A}^{--}(\tau_0) - \boldsymbol{A}^{--}(\tau_1))$$

which can be non-zero even when $\hat{A}_1$ is independent of time, because of the varying particle interpretation. Herein lies an elegant physical description of quantum anomalies, as outlined in [1]. A treatment of the axial anomaly, described in terms of this physical mechanism, has been given in [29] and in [17].

The derivation of $\langle F_{\tau_0}(\tau_1)|\hat{A}_{\mathrm{phys}}(\tau_1)|F_{\tau_0}(\tau_1)\rangle$ for an arbitrary state $|F_{\tau_0}(\tau_1)\rangle$ is identical to the derivation of (31), with result:

$$\begin{aligned}\langle \tbinom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}_{\tau_0}(\tau_1)|\hat{A}_{\mathrm{phys}}(\tau_1)|\tbinom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}_{\tau_0}(\tau_1)\rangle \\ = \sum_{k=1}^{m} \langle u_{i_k,\tau_0}(\tau_1)|\hat{A}_1(\tau_1)|u_{i_k,\tau_0}(\tau_1)\rangle - \sum_{k=1}^{n} \langle v_{j_k,\tau_0}(\tau_1)|\hat{A}_1(\tau_1)|v_{j_k,\tau_0}(\tau_1)\rangle \\ + \langle \mathrm{vac}_{\tau_0}(\tau_1)|\hat{A}_{\mathrm{phys}}(\tau_1)|\mathrm{vac}_{\tau_0}(\tau_1)\rangle \qquad (33)\end{aligned}$$

We now consider some simple examples of physical extension.

The charge operator is easily recognised as the physical extension of the unit operator on $\mathcal{H}$, $\hat{Q} = e\hat{1}_{\mathrm{phys}}$. This is appropriate, since the norm $\langle\psi|\hat{1}_1|\psi\rangle$ of a state $\psi \in \mathcal{H}$ is the Noether charge of the Dirac Lagrangian that is conjugate to changes in phase. Charge conservation now follows directly from the fact that evolution is grade-preserving.

The operator $\hat{N}_{\mathrm{phys}}(\tau)$ that represents the number of particles (including antiparticles) present at time τ is the physical extension of $\hat{N}_1(\tau) = \hat{P}_\tau^+ - \hat{P}_\tau^-$, where $\hat{P}_\tau^\pm : \mathcal{H} \to \mathcal{H}^\pm(\tau)$ are the projection operators onto $\mathcal{H}^\pm(\tau)$. Clearly $\hat{N}_1(\tau_0)$ commutes with $\hat{H}_1(\tau_0)$, but does not in general commute with time evolution (since it does not commute with $\hat{H}_1(\tau)$ for $\tau \neq \tau_0$). $\hat{N}_{\mathrm{phys}}(\tau_0)$ inherits both of these properties. Therefore the number operator $\hat{N}_{\mathrm{phys}}(\tau_0)$ represents a well-defined physical observable, but one which is not conserved. When acting on states in standard form, it gives:

$$\hat{N}_{\mathrm{phys}}(\tau_0)|\tbinom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}_{\tau_0}\rangle = (m+n)|\tbinom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}_{\tau_0}\rangle \qquad (34)$$

so that it is positive definite and has integer eigenvalues. From (31), the expectation value of $\hat{N}_{\mathrm{phys}}(\tau_1)$ in the 'evolved vacuum' $|\mathrm{vac}_{\tau_0}(\tau_1)\rangle$ is given

by:

$$\begin{aligned} N_{\mathrm{vac},\tau_0}(\tau_1) &\equiv \langle \mathrm{vac}_{\tau_0}(\tau_1)|\hat{N}_{\mathrm{phys}}(\tau_1)|\mathrm{vac}_{\tau_0}(\tau_1)\rangle \\ &= \mathrm{Trace}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger + \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger) \\ &= \sum_i \{N^+_{i,\tau_0}(\tau_1) + N^-_{i,\tau_0}(\tau_1)\} \end{aligned} \tag{35}$$

where $N^+_{i,\tau_0}(\tau_1) = (\boldsymbol{\beta}\boldsymbol{\beta}^\dagger)_{ii} = \sum_j |\langle u_{i,\tau_1}|v_{j,\tau_0}(\tau_1)\rangle|^2$ is the expectation value of the physical extension of $|u_{i,\tau_1}\rangle\langle u_{i,\tau_1}|$, and represents the probability that the degree of freedom u_{i,τ_1} is occupied in $|\mathrm{vac}_{\tau_0}(\tau_1)\rangle$, i.e. that particle i is present. $N^-_{i,\tau_0}(\tau_1) = (\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{ii} = \sum_j |\langle v_{i,\tau_1}|u_{j,\tau_0}(\tau_1)\rangle|^2$ is the expectation value of the physical extension of $-|v_{i,\tau_1}\rangle\langle v_{i,\tau_1}|$; it represents the probability that the degree of freedom v_{i,τ_1} is unoccupied in $|\mathrm{vac}_{\tau_0}(\tau_1)\rangle$, i.e. that antiparticle i is present. The Bogoliubov conditions imply $\mathrm{Trace}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger) = \mathrm{Trace}(\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)$, which again expresses charge conservation. By defining the projection operators $\hat{P}^\pm_{\tau_0}(\tau)$ by:

$$\hat{P}^+_{\tau_0}(\tau) \equiv \sum_i |u_{i,\tau_0}(\tau)\rangle\langle u_{i,\tau_0}(\tau)| = \hat{U}_1(\tau,\tau_0)\hat{P}^+_{\tau_0}\hat{U}_1^\dagger(\tau,\tau_0) \tag{36}$$

$$\hat{P}^-_{\tau_0}(\tau) \equiv \sum_i |v_{i,\tau_0}(\tau)\rangle\langle v_{i,\tau_0}(\tau)| = \hat{U}_1(\tau,\tau_0)\hat{P}^-_{\tau_0}\hat{U}_1^\dagger(\tau,\tau_0) \tag{37}$$

we can write $N^\pm_{i,\tau_0}(\tau_1)$ as:

$$N^+_{i,\tau_0}(\tau_1) = \langle u_{i,\tau_1}|\hat{P}^-_{\tau_0}(\tau_1)|u_{i,\tau_1}\rangle \qquad N^-_{i,\tau_0}(\tau_1) = \langle v_{i,\tau_1}|\hat{P}^+_{\tau_0}(\tau_1)|v_{i,\tau_1}\rangle \tag{38}$$

We can rewrite (31) in terms of $\hat{P}^\pm_{\tau_0}(\tau)$ as:

$$\begin{aligned} \langle \mathrm{vac}_{\tau_0}(\tau_1)|\hat{\mathrm{A}}_{\mathrm{phys}}(\tau_1)|\mathrm{vac}_{\tau_0}(\tau_1)\rangle &= \mathrm{Trace}(\hat{\mathrm{A}}_1(\tau_1)(\hat{\mathrm{P}}^-_{\tau_0}(\tau_1) - \hat{\mathrm{P}}^-_{\tau_1})) \\ &= -\mathrm{Trace}(\hat{\mathrm{A}}_1(\tau_1)(\hat{\mathrm{P}}^+_{\tau_0}(\tau_1) - \hat{\mathrm{P}}^+_{\tau_1})) \end{aligned} \tag{39}$$

where we have used $\hat{P}^+_{\tau_0}(\tau_1) + \hat{P}^-_{\tau_0}(\tau_1) = \hat{1} = \hat{P}^+_{\tau_1} + \hat{P}^-_{\tau_1}$. This result can be generalised to any state $|F\rangle$ of the form $|F(\tau)\rangle = |\psi_1(\tau)\wedge\psi_2(\tau)\cdots\wedge\psi_n(\tau)\rangle$ where $\psi_1(\tau),\psi_2(\tau),\ldots\psi_n(\tau)$ are orthonormal (which includes all $|\binom{i_1 i_2\ldots i_m}{j_1 j_2\ldots j_n}_{\tau_0}(\tau)\rangle$). In this case, define:

$$\hat{P}^-_{|F\rangle}(\tau) \equiv \sum_{i\in I} |\psi_i(\tau)\rangle\langle\psi_i(\tau)|$$

$$\text{and } \hat{P}^+_{|F\rangle}(\tau) = \equiv \sum_{i\notin I} |\psi_i(\tau)\rangle\langle\psi_i(\tau)| = \hat{1} - \hat{P}^-_{|F\rangle}(\tau)$$

where $\sum_{i\in I}$ runs over $\{\psi_1,\dots\psi_n\}$, and $\sum_{i\notin I}$ runs over the orthogonal complement of this. Then (33) can be written as:

$$\begin{aligned}\langle F(\tau)|\hat{A}_{\mathrm{phys}}(\tau)|F(\tau)\rangle &= \mathrm{Trace}(\hat{\mathrm{A}}_1(\tau)(\hat{\mathrm{P}}^-_{|\mathrm{F}\rangle}(\tau)-\hat{\mathrm{P}}^-_\tau))\\ &= -\mathrm{Trace}(\hat{\mathrm{A}}_1(\tau)(\hat{\mathrm{P}}^+_{|\mathrm{F}\rangle}(\tau)-\hat{\mathrm{P}}^+_\tau))\\ &= \mathrm{Trace}(\hat{\mathrm{A}}_1(\tau)(\hat{\mathrm{P}}^-_{|\mathrm{F}\rangle}(\tau)\hat{\mathrm{P}}^+_\tau-\hat{\mathrm{P}}^+_{|\mathrm{F}\rangle}(\tau)\hat{\mathrm{P}}^-_\tau))\end{aligned} \tag{40}$$

The relation of these projection operators to the various Greens functions of the theory, and to the first order 'Dirac density matrix' of multiparticle quantum mechanics (see for instance [30], pp 9-10) is described in the Appendix.

3.4. Fluctuations

Fluctuations in expectation values can be calculated using (13). In the notation of the previous subsection, we can write (13 as:

$$\begin{aligned}\langle F(\tau)|(\hat{A}_{\mathrm{phys}})^2|F(\tau)\rangle-(\langle F(\tau)|\hat{A}_{\mathrm{phys}}|F(\tau)\rangle)^2 &= \sum_{i\in I,j\notin I}|\langle\psi_i(\tau)|\hat{A}_1|\psi_j(\tau)\rangle|^2\\ = \mathrm{Trace}(\hat{P}^-_{|F\rangle}(\tau)\hat{A}_1(\tau)\hat{P}^+_{|F\rangle}(\tau)\hat{A}_1(\tau)) &= -\tfrac{1}{2}\mathrm{Trace}([\hat{\mathrm{P}}^-_{|\mathrm{F}\rangle}(\tau),\hat{\mathrm{A}}_1(\tau)]^2)\end{aligned}$$

In the evolved vacuum $|\mathrm{vac}_{\tau_0}(\tau)\rangle$ this becomes:

$$\begin{aligned}\langle\hat{A}^2\rangle &\equiv \langle\mathrm{vac}_{\tau_0}(\tau)|(\hat{A}_{\mathrm{phys}})^2|\mathrm{vac}_{\tau_0}(\tau)\rangle-\langle\mathrm{vac}_{\tau_0}(\tau)|\hat{A}_{\mathrm{phys}}|\mathrm{vac}_\tau(\tau)\rangle^2\\ &= \sum_{i,j}|\langle v_{i,\tau_0}(\tau)|\hat{A}_1(\tau)|u_{j,\tau_0}(\tau)\rangle|^2\\ &= \mathrm{Trace}(\hat{P}^-_{\tau_0}(\tau)\hat{A}_1(\tau)\hat{P}^+_{\tau_0}(\tau)\hat{A}_1(\tau))\\ &= \mathrm{Trace}(\boldsymbol{A}_{\mathrm{F}}\boldsymbol{A}_{\mathrm{F}}^\dagger)\end{aligned} \tag{41}$$

$$\text{where } \boldsymbol{A}_F\equiv\boldsymbol{\beta}^\dagger\boldsymbol{A}^{++}\boldsymbol{\alpha}+\boldsymbol{\epsilon}^\dagger\boldsymbol{A}^{--}\boldsymbol{\gamma}+\boldsymbol{\beta}^\dagger\boldsymbol{A}^{+-}\boldsymbol{\gamma}+\boldsymbol{\epsilon}^\dagger\boldsymbol{A}^{-+}\boldsymbol{\alpha} \tag{42}$$

Consider for example fluctuations in $\hat{N}^+(\tau)$, the total number of particles (not including antiparticles) in the evolved vacuum. Now $\hat{N}^+_1(\tau)=\hat{P}^+_\tau$ so that $\boldsymbol{A}_F=\boldsymbol{\beta}^\dagger\boldsymbol{\alpha}$, and:

$$\langle(\hat{N}^+)^2\rangle=\mathrm{Trace}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger(1-\boldsymbol{\beta}\boldsymbol{\beta}^\dagger)) \tag{43}$$

Consider a spatially uniform case, with $\beta_{\boldsymbol{p}\lambda;\boldsymbol{q}\sigma}(\tau,\tau_0)=\beta_{\boldsymbol{p}}\delta_{\lambda\sigma}(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q})$. Then:

$$\frac{N^+_{\text{vac},\tau_0}(\tau)}{V}=2\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}|\beta_{\boldsymbol{p}}|^2 \tag{44}$$

while
$$\langle\left(\frac{N^+_{\text{vac},\tau_0}(\tau)}{V}\right)^2\rangle=\frac{2}{V}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}|\beta_{\boldsymbol{p}}|^2(1-|\beta_{\boldsymbol{p}}|^2) \tag{45}$$

This result suggests that for large volumes, and for $|\beta_{\boldsymbol{p}}|^2<<1$, the fluctuations in the average particle density in a volume V vary inversely with V, and that in a given volume the rms error in the total number of particles is proportional to the square root of the number of particles, as would be expected for a weakly interacting system. The same result holds for the total number of antiparticles, or for the total number of pairs. (The fluctuations in the total charge are zero, due to the Bogoliubov conditions.) However, 'V' in (44) and (45) is formally infinite; $V=(2\pi)^3\delta^3(\underline{0})$. Before we can confirm our interpretation of (44) and (45) we must consider finite volume measurements.

3.5. Finite Volume Measurements

To measure the quantity A in a volume V (on Σ_τ) it seems reasonable to consider the operator $\hat{A}_{V,\text{phys}}$ obtained by Hermitian extension and vacuum subtraction from the operator:

$$\hat{A}_{V,1}\equiv\tfrac{1}{2}(\hat{\theta}_V\hat{A}_1+\hat{A}_1\hat{\theta}_V)\text{where }\hat{\theta}_V\psi(x_\tau)\equiv\begin{cases}\psi(x_\tau) & x_\tau\in V\\ 0 & x_\tau\notin V\end{cases} \tag{46}$$

where x_τ is shorthand for $x|_{\Sigma_\tau}$ (the restriction of x onto Σ_τ). The matrix elements of $\hat{A}_{V,1}$ are related to those of $\hat{A}_1$ by restricting the volume integral to V, taking the appropriate combination of surface terms to ensure Hermiticity. In the Canonical approach, $\hat{A}_{V,H}$ is similarly obtained by restricting the bilinear $\int_{\Sigma_\tau}e(x)\frac{1}{2}[\overline{\hat{A}_1\hat{\psi}(x)}\gamma^\mu(x)\hat{\psi}(x)+\overline{\hat{\psi}(x)}\gamma^\mu(x)\hat{A}_1\hat{\psi}(x)]d\Sigma_\mu$ to the volume V (here $\hat{\psi}(x)$ is the field operator).

As defined, however, $\hat{A}_{V,\text{phys}}$ gives rise to some serious problems:

1. The fluctuations in $\hat{N}^+_{V,\text{phys}}(\tau)$, $\hat{N}^-_{V,\text{phys}}(\tau)$, $\hat{Q}_{V,\text{phys}}$ and $\hat{H}_{V,\text{phys}}(\tau)$ are infinite for any finite V, even in the physical vacuum $|\text{vac}_\tau\rangle$, and even for

an inertial observer in flat empty space! To see this, note that:

$$\langle \mathrm{vac}_\tau|(\hat{N}^{\pm}_{V,\mathrm{phys}})^2|\mathrm{vac}_\tau\rangle = \sum_{ij}|\langle u_{j,\tau}|v_{i,\tau}\rangle_V|^2 \tag{47}$$

$$\langle \mathrm{vac}_\tau|(\hat{H}_{V,\mathrm{phys}})^2|\mathrm{vac}_\tau\rangle = \sum_{ij}(E^+_j - E^-_i)^2|\langle u_{j,\tau}|v_{i,\tau}\rangle_V|^2 \tag{48}$$

where $\langle u_{j,\tau}|v_{i,\tau}\rangle_V$ signifies that the integral in the inner product has been restricted to V (it would be zero otherwise). For convenience $|u_{j,\tau}\rangle$ is chosen to be an eigenstate with eigenvalue E^+_j and $|v_{i,\tau}\rangle$ is an eigenstate with eigenvalue $-E^-_i$. For an inertial observer in flat 1+1 dimensional space, and V being the region $|x| < \frac{L}{2}$, this becomes:

$$\langle \mathrm{vac}_\tau|(\hat{H}_{V,\mathrm{phys}})^2|\mathrm{vac}_\tau\rangle = \int \frac{dp}{2\pi}\frac{dq}{2\pi}F(p,q)$$

$$\text{where } F(p,q) \equiv (E_p - E_q)^2\frac{(E_p+p)(E_q+q)}{2E_p 2E_q}(1-\frac{m^2}{(E_p+p)(E_q+q)})^2\frac{\sin^2[(p+q)L]}{(p+q)^2}$$

which is easily seen to diverge (the divergence is just as bad in 3+1 dimensions).

2. The expectation values of $\hat{N}^{\pm}_{V,\mathrm{phys}}(\tau)$ in the evolved vacuum $|\mathrm{vac}_{\tau_0}(\tau)\rangle$ are given by:

$$\langle \mathrm{vac}_{\tau_0}(\tau)|\hat{\mathrm{N}}^{+}_{\mathrm{V,phys}}(\tau)|\mathrm{vac}_{\tau_0}(\tau)\rangle = \mathrm{Trace}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger\langle \mathrm{u}_{\mathrm{i},\tau}|\mathrm{u}_{\mathrm{j},\tau}\rangle_{\mathrm{V}} + \Re(\boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger\langle \mathrm{u}_{\mathrm{i},\tau}|\mathrm{v}_{\mathrm{j},\tau}\rangle_{\mathrm{V}})) \tag{49}$$

$$\langle \mathrm{vac}_{\tau_0}(\tau)|\hat{\mathrm{N}}^{-}_{\mathrm{V,phys}}(\tau)|\mathrm{vac}_{\tau_0}(\tau)\rangle = \mathrm{Trace}(\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger\langle \mathrm{v}_{\mathrm{i},\tau}|\mathrm{v}_{\mathrm{j},\tau}\rangle_{\mathrm{V}} - \Re(\boldsymbol{\epsilon}\boldsymbol{\beta}^\dagger\langle \mathrm{u}_{\mathrm{i},\tau}|\mathrm{v}_{\mathrm{j},\tau}\rangle_{\mathrm{V}})) \tag{50}$$

The first terms represent the obvious contribution from the sum over created particles of the probability distribution of each created particle. The final term has no obvious interpretation. It is comparable in magnitude to the first term, but is not positive definite, and it affects $\langle \mathrm{vac}_{\tau_0}(\tau)|\hat{\mathrm{N}}^{+}_{\mathrm{V,phys}}(\tau)|\mathrm{vac}_{\tau_0}(\tau)\rangle$ with opposite sign to $\langle \mathrm{vac}_{\tau_0}(\tau)|\hat{\mathrm{N}}^{-}_{\mathrm{V,phys}}(\tau)|\mathrm{vac}_{\tau_0}(\tau)\rangle$.

All these problems stem from the fact that $\hat{\theta}_V$ does not commute with $\hat{P}^{\pm}_\tau$. This is the well-known fact that particle states cannot be confined to a finite volume without introducing a negative-energy component. Consequently, even if $[\hat{A}_1, \hat{P}^{\pm}_\tau] = 0$ the same will not be true of $\hat{A}_{V,1}$. Hence, even if $|\mathrm{vac}_\tau\rangle$ is an eigenstate of $\hat{A}_{\mathrm{phys}}$, it will not in general be an eigenstate of $\hat{A}_{V,\mathrm{phys}}$ for finite V. This can be seen in the Canonical approach by considering the transition from operators expressed in terms of $\hat{\psi}(x)$ to operators

expressed in terms of creation/anihilation operators. The absence of terms proportional to $a_i^\dagger b_j^\dagger$ relies on the relation $\langle u_i|\hat{A}_1|v_j\rangle = 0$. However, when the integral is restricted to a finite volume this matrix element can be non-zero even if $|v_j\rangle$ is an eigenstate of $\hat{A}_1$. It is these finite volume overlaps between positive and negative energy states that contribute to (47) and (48), and lead to the unphysical second term in (49) and (50).

This problem is easily overcome. Define:

$$\hat{\theta}_V^\tau \equiv \hat{P}_\tau^+ \hat{\theta}_V \hat{P}_\tau^+ + \hat{P}_\tau^- \hat{\theta}_V \hat{P}_\tau^- \tag{51}$$

$$\text{and } \hat{A}_{V,1} \equiv \tfrac{1}{2}(\hat{\theta}_V^\tau \hat{A}_1 + \hat{A}_1 \hat{\theta}_V^\tau) \tag{52}$$

The expressions considered in (46) - (50) will henceforth be denoted $\hat{A}_{V,1}^{\text{naive}}$, $\hat{A}_{V,\text{phys}}^{\text{naive}}$, etc. $\hat{A}_{V,1}$ now has the property that $[\hat{A}_{V,1}, \hat{P}_\tau^\pm] = 0$ (for all V) if and only if $[\hat{A}_1, \hat{P}_\tau^\pm] = 0$. Hence, if $|\text{vac}_\tau\rangle$ is an eigenstate of $\hat{A}_{\text{phys}}$, then it is an eigenstate of $\hat{A}_{V,\text{phys}}$ for all V, so that the fluctuations of A in any finite volume in the physical vacuum $|\text{vac}_\tau\rangle$ will be zero. This applies for instance to $\hat{N}^\pm(\tau)$, $\hat{Q}$ and $\hat{H}$, and resolves problem 1 above.

If $[\hat{A}_1, \hat{P}_\tau^\pm] = 0$, then:

$$\langle \text{vac}_{\tau_0}(\tau_1)|\hat{A}_{V,\text{phys}}(\tau_1)|\text{vac}_{\tau_0}(\tau_1)\rangle = \text{Trace}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger \boldsymbol{A}_V^{++} - \boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger \boldsymbol{A}_V^{--})$$

where $\boldsymbol{A}_{V,jk}^{++} \equiv \langle u_{j,\tau_1}|\hat{A}_{1,V}^{\text{naive}}(\tau_1)|u_{k,\tau_1}\rangle$ and $\boldsymbol{A}_{jk}^{--} \equiv \langle v_{j,\tau_1}|\hat{A}_{1,V}^{\text{naive}}(\tau_1)|v_{k,\tau_1}\rangle$. The terms describing finite-volume overlaps between positive and negative energy states have been removed, resolving problem 2. For instance, we can now write:

$$N_{i,V,\tau_0}^-(\tau) = \Re(\langle v_{i,\tau}|\hat{\theta}_V \hat{P}_\tau^- \hat{P}_{\tau_0}^+(\tau)|v_{i,\tau}\rangle = \sum_k \Re[\text{Trace}((\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{\text{ik}} \langle \text{v}_{\text{k},\tau}|\text{v}_{\text{i},\tau}\rangle_{\text{V}})]$$

$$N_{i,V,\tau_0}^+(\tau) = \Re(\langle u_{i,\tau}|\hat{\theta}_V \hat{P}_\tau^+ \hat{P}_{\tau_0}^-(\tau)|u_{i,\tau}\rangle = \sum_k \Re[\text{Trace}((\boldsymbol{\beta}\boldsymbol{\beta}^\dagger)_{\text{ik}} \langle \text{u}_{\text{k},\tau}|\text{u}_{\text{i},\tau}\rangle_{\text{V}})]$$

from which it follows that:

$$\begin{aligned}
N^-_{V,\tau_0}(\tau) &= \mathrm{Trace}(\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger\langle \mathrm{v}_{\mathrm{k},\tau}|\mathrm{v}_{\mathrm{i},\tau}\rangle_{\mathrm{V}}) \\
&= \int_V e(x) J^{-\mu}_{\tau_0}(x) d\Sigma_\mu \quad = \int_V n^-_{\tau_0}(\boldsymbol{x}) d^3\boldsymbol{x} \qquad (53)
\end{aligned}$$

where $$J^{-\mu}_{\tau_0}(x) \equiv \sum_{ik} (\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{ik} \bar{v}_{k,\tau(x)}(x)\gamma^\mu v_{i,\tau(x)}(x)$$

and $$n^-_{\tau_0}(\boldsymbol{x}) \equiv e(x_\tau) J^{-\mu}_{\tau_0}(x_\tau) \frac{\partial_\mu \tau}{\partial_0 \tau}|_{x=x_\tau} \text{ where } x^\mu_\tau = (t_\tau(\boldsymbol{x}), \boldsymbol{x})$$

$$\begin{aligned}
N^+_{V,\tau_0}(\tau) &= \mathrm{Trace}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger\langle u_{k,\tau}|u_{i,\tau}\rangle_V) \\
&= \int_V e(x) J^{+\mu}_{\tau_0}(x) d\Sigma_\mu \quad = \int_V n^+_{\tau_0}(x) d^3\boldsymbol{x} \qquad (54)
\end{aligned}$$

where $$J^{+\mu}_{\tau_0}(x) \equiv \sum_{ik} (\boldsymbol{\beta}\boldsymbol{\beta}^\dagger)_{ik} \bar{u}_{k,\tau(x)}(x)\gamma^\mu u_{i,\tau(x)}(x)$$

and $$n^+_{\tau_0}(\boldsymbol{x}) \equiv e(x_\tau) J^{+\mu}_{\tau_0}(x_\tau) \frac{\partial_\mu \tau}{\partial_0 \tau}|_{x=x_\tau}$$

Then $N_{V,\tau_0}(\tau) = \int_V e(x) J^{+,\mu}_{\tau_0}(x) + J^{-,\mu}_{\tau_0}(x) d\Sigma_\mu$ and $Q_{V,\tau_0}(\tau) = e\int_V e(x) J^{+,\mu}_{\tau_0}(x) - J^{-,\mu}_{\tau_0}(x) d\Sigma_\mu$. The fluctuations in $N^+_{V,\tau_0}(\tau)$ are given by:

$$\langle(\hat{N}^+_V)^2\rangle = \mathrm{Trace}((\boldsymbol{\beta}\boldsymbol{\beta}^\dagger)_{\mathrm{ij}}\langle \mathrm{u}_{\mathrm{j},\tau}|\mathrm{u}_{\mathrm{k},\tau}\rangle_{\mathrm{V}}(1-\boldsymbol{\beta}\boldsymbol{\beta}^\dagger)_{\mathrm{kl}}\langle \mathrm{u}_{\mathrm{l},\tau}|\mathrm{u}_{\mathrm{m},\tau}\rangle_{\mathrm{V}}) \qquad (55)$$

which clearly reduces to (43) when V covers all of Σ_τ. In the spatially uniform case, with $\beta_{\boldsymbol{p}\lambda;\boldsymbol{q}\sigma}(\tau,\tau_0) = \beta_{\boldsymbol{p}}\delta_{\lambda\sigma}(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q})$, it is straightforward to show that (45) also holds approximately for large finite V, confirming the interpretation of (45) given at the end of Section 3.4. Similarly, the fluctuations in $N^-_{V,\tau_0}(\tau)$ are given by:

$$\langle(\hat{N}^-_V)^2\rangle = \mathrm{Trace}((\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{\mathrm{ij}}\langle \mathrm{v}_{\mathrm{j},\tau}|\mathrm{v}_{\mathrm{k},\tau}\rangle_{\mathrm{V}}(1-\boldsymbol{\gamma}\boldsymbol{\gamma}^\dagger)_{\mathrm{kl}}\langle \mathrm{v}_{\mathrm{l},\tau}|\mathrm{v}_{\mathrm{m},\tau}\rangle_{\mathrm{V}}) \qquad (56)$$

In the absence of an electromagnetic field, Charge Conjugation invariance implies that $N^+_{V,\tau_0}(\tau) = N^-_{V,\tau_0}(\tau)$ and $\langle(\hat{N}^+_V)^2\rangle = \langle(\hat{N}^+_V)^2\rangle$ for all V. Discrete symmetries will be addressed in a future publication.

We have described, both here and in previous publications [16, 14], the utility of a finite-time particle interpretation. In [6] we demonstrated with examples how it becomes possible to track the particle production process, finding not only how many particles are created, but also when they are created, and how they behave after their creation. Having now defined finite-volume operators which have well-defined fluctuations, we can also say (with controllable precision) where the particles are created. In a future publication we will show that this makes it possible to treat ‘particle

creation' and 'vacuum polarisation' effects within the same framework ([8] presents a similar treatment for electrostatic fields). In particular, for an electromagnetic potential step, modes with evanescent contributions give rise to a nonzero charge distribution in the vicinity of the barrier (on a length scale $\lambda_c = \frac{\hbar}{mc}$), providing a vacuum polarisation which partially screens the barrier. Meanwhile, the 'Klein modes', present only for a potential step larger than $2mc^2$, correspond to the standard 'particle creation' effect, with created plane wave states (persisting to spatial infinity). Conventional 'tunneling' methods, based on calculating transmission/reflection coefficients, describe only those modes which persist as plane waves either side of the barrier. Methods based only on $\hat{N} = \sum_i \{a_i^\dagger a_i + b_i^\dagger b_i\}$, pick up the total number of created particles, but this generally must be divided by the infinite spatial volume $(2\pi)^3 \delta(\underline{0})$ to obtain an average particle density. Again, only those modes contribute which persist to spatial infinity.

Further details of our approach to fermionic QFT are given in [1, 16]. We now study a simple example involving a particle horizon; the well-known Unruh effect, concerning a uniformly accelerating observer in flat Minkowski spacetime.

4. RINDLER SPACE: A SIMPLE HORIZON

The Unruh effect [31, 32] is one of the most studied and most cited examples of particle creation, and demonstrates more than any other the fact that the concept of particle is observer-dependent. Useful references are [7, 33, 8] and the further references therein. An early treatment of fermions in Rindler space is [34] (presented again in the same authors text [8]), while [35] presents a very thorough review of fermions in Rindler space. We now consider this problem using the present formalism. We begin by showing that the radar time of a uniformly accelerating observer is indeed Rindler time, and that the eigenstates of $\hat{H}_1$ are stationary states of Rindler time. This demonstrates the consistency of this approach with standard derivations. We complete the derivation in 1+1 dimensions for both massive and massless particles. Although the rest of this derivation is quite standard, we have also described the spatial distribution of Rindler particles, which has received little attention to date.

4.1. Radar Time for Accelerating Observers

Consider first the 1+1 dimensional case. We have

$$x^{\mu}_{(\gamma)}(\tau) \equiv (t(\tau), z(\tau)) = (\frac{\sinh(a\tau)}{a}, \frac{\cosh(a\tau)}{a}) \tag{57}$$

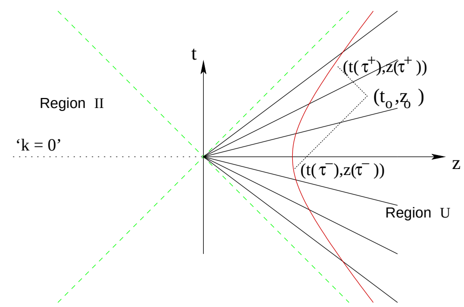

FIG. 2. Hypersurfaces of simultaneity of a uniformly accelerating observer.

Consider a point (t_o, z_o) to the right of the observer's worldline, as shown in figure 2. Clearly, the $\tau^{\pm}$ for this point must satisfy:

$$t(\tau^+) - t_o = z_o - z(\tau^+) \qquad t_o - t(\tau^-) = z_o - z(\tau^-) \tag{58}$$

From (57),

$$e^{a\tau^+} = a(z_o + t_o) \qquad e^{-a\tau^-} = a(z_o - t_o)$$

from which we deduce that $\tau = \frac{1}{2a}\log(\frac{z_o+t_o}{z_o-t_o})$. For points (t_o, z_o) to the left of the observers worldline (but still in Region U), the roles of τ^+ and τ^- are reversed, leaving τ unchanged. We can therefor drop the subscripts, and write:

$$\tau(x) = \frac{1}{2a}\log(\frac{z+t}{z-t}) \tag{59}$$

which is the Rindler time-coordinate, and covers only region U. The hypersurfaces Σ_{τ_0} are given by $t_{\tau_0}(z) = z\tanh(a\tau_0)$, as shown in figure 2. The radar distance $\rho(x)$ (which is positive by construction) is given by $\rho(x) = \frac{|\log(a^2(z^2-t^2))|}{2a} = \frac{|\log(a^2u^2)|}{2a}$, where $u \equiv \sqrt{z^2-t^2}$. In (τ, u) coordinates the metric takes the familiar form:

$$ds^2 = a^2u^2d\tau^2 - du^2$$

Two uniformly accelerating observers, each with different a, will have the same hypersurfaces of simultaneity, but will 'tick off' these hypersurfaces at different rates. The vector field k^μ is given by:

$$k = z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} \qquad \text{or } = \frac{\partial}{\partial \tau} \text{ in Rindler coordinates}$$

which is the Killing vector field that is used to define positive/negative frequency modes in conventional derivations of the Unruh effect.

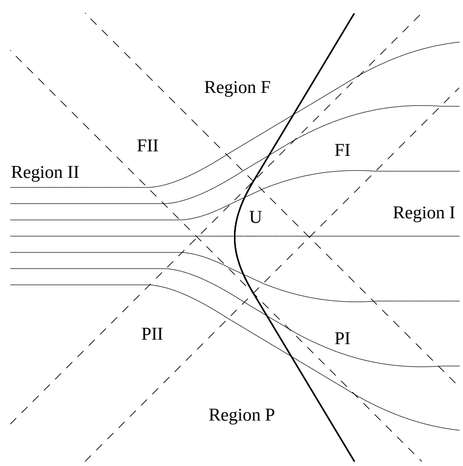

FIG. 3. Hypersurfaces of simultaneity of an observer undergoing uniform acceleration for a finite period of time (in Region U here) who is otherwise inertial.

To see the significance of the dotted line in region II, consider the 'Finite Acceleration Time' case shown in figure 3. In the limit as the 'Acceleration Time' approaches infinity, this case approaches that of a uniformly accelerating observer. In this limit the hypersurfaces of simultaneity (which are Cauchy) all approach this dotted line in region II ($k_\mu(x) \to 0$ there), and a particle horizon forms. This explains the requirement (essential to many derivations of the Unruh Effect [34, 8, 7, 36]) that the 'Rindler modes' must be those that are zero throughout region II. Further details of the 'Finite Acceleration Time' case can be found in [6].

Consider now the 3+1 dimensional case, with $x^{\mu}_{(\gamma)}(\tau) = (\frac{\sinh(a\tau)}{a}, 0, 0, \frac{\cosh(a\tau)}{a})$. In this case equations (58) can be replaced by (dropping subscripts)

$$(t(\tau^+) - t)^2 = (z - z(\tau^+))^2 + |\boldsymbol{x}_\perp|^2$$

(where $|\boldsymbol{x}_\perp|^2 \equiv x^2 + y^2$), which applies to both τ^+ and τ^-. Substitution of the expressions for $t(\tau)$ and $z(\tau)$ and rearrangement gives:

$$T_\pm^2 - \left(\frac{1}{a(z-t)} + a(z+t)\right) T_\pm + \frac{z+t}{z-t} = 0 \tag{60}$$

where $T_\pm \equiv e^{a\tau^\pm}$ are the two roots of (60). From this it follows immediately that $e^{2a\tau} = T_+ T_- = \frac{z+t}{z-t}$, just as in the 1+1 dimensional case. $k_\mu(x)$ is also as before, and we can again define $u \equiv \sqrt{z^2 - t^2}$ so that the metric can be written as

$$ds^2 = a^2 u^2 d\tau^2 - du^2 - dx^2 - dy^2$$

reproducing the Rindler coordinates.

We have now justified the use of Rindler coordinates for studying uniformly accelerating observers, and the derivation of the conventional Unruh effect becomes standard [8, 34, 35]. We reproduce the massive and massless cases here, restricting ourselves to 1+1 dimensions for convenience.

By defining $\gamma_0 = au\bar{\gamma}_0$ and $\gamma_3 = \bar{\gamma}_3$ (where $\bar{\gamma}_0, \bar{\gamma}_3$ are any matrices satisfying $\bar{\gamma}_0^2 = 1 = -\bar{\gamma}_3^2$ and $\{\bar{\gamma}_0, \bar{\gamma}_3\} = 0$) we find that $\Gamma_0 = \frac{1}{2}a\bar{\sigma}_3$ and $\Gamma_3 = 0$. The Dirac equation can be written in the form of (17) as:

$$i\frac{\partial\psi}{\partial\tau} + \frac{i}{2}a\bar{\sigma}_3 = au(-i\bar{\sigma}_3\frac{\partial\psi}{\partial u}) + au\bar{\gamma}^0\psi \tag{61}$$

The inner product in U is written in these coordinates as:

$$\langle\psi|\phi\rangle = \int_0^\infty \psi^\dagger\phi du \tag{62}$$

The $u\frac{\partial}{\partial u}$ term on the RHS of (61) implies, as expected, that $\hat{H}_{ev}$ is not Hermitian. A simple calculation reveals that $\hat{H}_{ev}^\dagger = \hat{H}_{ev} - ia\bar{\sigma}_3$. Hence (61) can be rewritten in terms of $\hat{H}_1$ as:

$$i\frac{\partial\psi}{\partial\tau} = \hat{H}_1\psi \equiv au(-i\bar{\sigma}_3\frac{\partial\psi}{\partial u}) + au\bar{\gamma}^0\psi - \frac{i}{2}a\bar{\sigma}_3 \tag{63}$$

It is easy to verify that (19) is indeed satisfied. The eigenstates of $\hat{H}_1$ are the stationary states, justifying again the choice made in previous derivations. Since the eigenstates of $\hat{H}_1$ are stationary, then although the Rindler observer disagrees that the Minkowski vacuum is empty, he still agrees that the particle content does not change with τ.

4.1.1. Solutions in Region U

Consider solutions of the form:

$$^{I}\psi_\omega(x) = (f(u)\phi_+ + g(u)\phi_-)\psi_\omega(u)e^{-i\omega\tau}$$

where $\phi_\pm$ are basis spinors, satisfying

$$\phi_\lambda^\dagger\phi_\sigma = 2\delta_{\lambda\sigma} \qquad \bar{\sigma}_3\phi_\pm = \pm\phi\pm \qquad \bar{\gamma}_0\phi_\pm = \phi\mp$$

The factor of 2 is for convenient comparison with the representation given in [8]. These are eigenstates of $\hat{H}_1$ for all τ, with eigenvalue ω.

Substitution into (63) gives:

$$\begin{aligned} \frac{i}{u}\left(u\frac{d}{du} + \tfrac{1}{2} - i\frac{\omega}{a}\right) f(u) &= mg(u) \\ \frac{-i}{u}\left(u\frac{d}{du} + \tfrac{1}{2} + i\frac{\omega}{a}\right) g(u) &= mf(u) \end{aligned} \tag{64}$$

from which we find that

$$(u\frac{d}{du}u\frac{d}{du})f(u) = \left[m^2u^2 - (\frac{\omega}{a} + \frac{i}{2})^2\right] f(u) \tag{65}$$

$$(u\frac{d}{du}u\frac{d}{du})g(u) = \left[m^2u^2 - (\frac{\omega}{a} - \frac{i}{2})^2\right] g(u) \tag{66}$$

Equations (65) and (66) can be identified with equation (21.66) of [8], or with page 4 of [37]. The only normalisable solution is of the form:

$$^{I}\psi_\omega(\tau,u) = \{H^{(1)}_{\frac{i\omega}{a}-\frac{1}{2}}(imu)\phi_+ + H^{(1)}_{\frac{i\omega}{a}+\frac{1}{2}}(imu)\phi_-\}e^{-iw\tau}$$

where $H^{(1)}_\nu(z)$ are Hankel Functions. These states satisfy [8]:

$$\langle^{I}\psi_\omega|^{I}\psi_{\omega'}\rangle = \frac{16a\delta(\omega-\omega')}{m(1+e^{\frac{-2\pi\omega}{a}})}$$

We shall return to these solutions after finding a convenient representation for the Minkowski modes.

4.2. The Minkowski modes

We have just found a basis for $\mathcal{H}_R^+$ and $\mathcal{H}_R^-$. Together they span all those modes that can be seen by the Rindler observer. They span only half of $\mathcal{H}_M$, since they do not include any states defined in region II. However, they are sufficient to calculate the Rindler number operator $\hat{N}_{1,R} = \hat{P}_R^+ - \hat{P}_R^-$, from which we can deduce the number of Rindler particles present in the Minkowski vacuum, along with their frequency and spatial distributions. To define the Minkowski vacuum, we could use the ordinary plane wave basis for Minkowski modes. However, it is more convenient for the massive case to use an alternative representation of Minkowski modes [8, 35, 34]. As a consistency check, we shall rederive the massless limit using Minkowski plane wave states.

The representation of Minkowski modes used in [8] (often called the 'Rindler Basis') is of the form:

$$u_\omega^M = N({}^I\psi_\omega - e^{\frac{\pi\omega}{2a}}\,{}^F\psi_\omega^{(+)} + e^{\frac{-\pi\omega}{2a}}\,{}^P\psi_\omega^{(+)} - ie^{\frac{-\pi\omega}{a}}\,{}^{II}\psi_\omega) \tag{67}$$

$$v_\omega^M = N(e^{\frac{-\pi\omega}{a}}\,{}^I\psi_\omega + e^{\frac{-\pi\omega}{2a}}\,{}^F\psi_\omega^{(-)} + e^{\frac{\pi\omega}{2a}}\,{}^P\psi_\omega^{(-)} + i\,{}^{II}\psi_\omega) \tag{68}$$

where ${}^{II}\psi_\omega(x) = \bar{\sigma}_3^I\psi_\omega(-x)$ is a state defined in Region II, and ${}^F\psi_\omega^{(\pm)}, {}^P\psi_\omega^{(\pm)}$ are states defined in Regions F and P respectively (see [8, 34] for details). These various states are defined only in their particular region, so the Dirac operator acting on any of these states does not give zero, but gives a distribution on the light cone through the origin. The coefficients in the above expansion are chosen to cancel these distributions. The $u_\omega^M(x)$ then form an orthogonal basis for $\mathcal{H}_M^+$, and the $v_\omega^M(x)$ form an orthogonal basis for $\mathcal{H}_M^-$ (irrespective of the sign of ω).

In (67) and (68), the states ${}^{II}\psi_\omega, {}^I\psi_\omega, {}^F\psi_\omega^{(\pm)}, {}^P\psi_\omega^{(\pm)}$ are not normalised. By rewriting (67) and (68) in terms of states ${}^{II}\psi_\omega, {}^I\psi_\omega$ which are each normalised to $2\pi\delta(\omega-\omega')$, and choosing N such that $u_\omega^M(x)$ and $v_\omega^M(x)$ are normalised to $2\pi\delta(\omega-\omega')$, equations (67) and (68) become:

$$u_\omega^M = \frac{1}{\sqrt{1+e^{\frac{-2\pi\omega}{a}}}}({}^I\psi_\omega^{\mathrm{norm}} - ie^{-\pi\omega}\,{}^{II}\psi_\omega^{\mathrm{norm}} + \text{ F, P, terms})$$

$$v_\omega^M = \frac{1}{\sqrt{1+e^{\frac{-2\pi\omega}{a}}}}(e^{-\pi\omega}\,{}^I\psi_\omega^{\mathrm{norm}} + i\,{}^{II}\psi_\omega^{\mathrm{norm}} + \text{ F, P, terms})$$

from which we can immediately extract the relations:

$$\alpha^I_{\omega\omega'} \equiv \langle {}^I\psi_\omega | u^M_{\omega'} \rangle = \frac{2\pi\delta(\omega-\omega')}{\sqrt{1+e^{\frac{-2\pi\omega}{a}}}} \tag{69}$$

$$\beta^I_{\omega\omega'} \equiv \langle {}^I\psi_\omega | v^M_{\omega'} \rangle = \frac{2\pi\delta(\omega-\omega')}{\sqrt{1+e^{\frac{2\pi\omega}{a}}}} \tag{70}$$

$$\gamma^I_{\omega\omega'} \equiv \langle {}^I\psi_{-\omega} | u^M_{\omega'} \rangle = \frac{2\pi\delta(\omega+\omega')}{\sqrt{1+e^{\frac{2\pi\omega}{a}}}} \tag{71}$$

$$\epsilon^I_{\omega\omega'} \equiv \langle {}^I\psi_{-\omega} | v^M_{\omega'} \rangle = \frac{2\pi\delta(\omega+\omega')}{\sqrt{1+e^{\frac{-2\pi\omega}{a}}}} \tag{72}$$

where ω is restricted to $\omega > 0$, but ω' can take any sign. Alternatively, these coefficients can be extracted using the observation that $\frac{u^M_\omega}{\sqrt{1+e^{\frac{-2\pi\omega}{a}}}} + \frac{v^M_\omega}{\sqrt{1+e^{\frac{2\pi\omega}{a}}}}$ (equal to ${}^I\psi^{\text{norm}}_\omega +$ F, P, terms) is the only linear combination of u^M_ω and v^M_ω that is zero in Region II. Although the inner products in (69) - (72) have been evaluated on a hypersurface (from (62)) which is not Cauchy in Minkowski space (it covers space according to the Rindler observer), it is still consistent with any inner product in Minkowski space, since the Rindler modes are all zero in region II.

From (69) - (72) it follows that:

$$N^\pm_\omega = \text{Trace}(\beta\beta^\dagger) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \frac{2\pi\delta(\omega\pm\omega')}{1+\mathrm{e}^{\frac{2\pi\omega}{\mathrm{a}}}} = \frac{1}{1+\mathrm{e}^{\frac{2\pi\omega}{\mathrm{a}}}} \tag{73}$$

which represents a thermal spectrum at Temperature $T = \frac{a}{2\pi k_B}$, as expected.

4.2.1. The Massless Case

In the massless case equations (64) decouple, and we find two independent solutions for each ω,

$$|\psi_{\omega,1}\rangle = \phi_+(au)^{\frac{i\omega}{a}-\frac{1}{2}} e^{-i\omega\tau} \text{ and } |\psi_{\omega,2}\rangle = \phi_-(au)^{\frac{-i\omega}{a}-\frac{1}{2}} e^{-i\omega\tau} \tag{74}$$

Each of these is normalised to $\langle \psi_{\omega,\sigma} | \psi_{\omega',\sigma'} \rangle = (2\pi)\delta_{\sigma\sigma'}\delta(\omega-\omega')$. The massless limit of $|{}^I\psi^{\text{norm}}_\omega\rangle$ would yield one specific linear combination of $|\psi_{\omega,1}\rangle$ and $|\psi_{\omega,2}\rangle$. However, we shall find that $|\psi_{\omega,1}\rangle$ and $|\psi_{\omega,2}\rangle$ each lead to a thermal spectrum, so that we need not restrict ourselves to this one linear combination.

The plane wave states $\mathcal{H}_M^{\pm}$ also take a particularly simple form in the massless case. A basis of $\mathcal{H}_M^{+}$ is provided by states of the form:

$$\phi_+ e^{-ip(t-x)} \text{ and } \phi_- e^{-ip(t+x)} \text{ for } p>0$$

and a basis for $\mathcal{H}_M^{-}$ is provided by states of the form:

$$\phi_+ e^{ip(t-x)} \text{ and } \phi_- e^{ip(t+x)} \text{ for } p>0$$

This allows us to write:

$$N_{\omega,1}^{+} = \sum_p |\beta_{\omega,1p}|^2 = \frac{1}{L}\int_0^\infty \frac{dp}{2\pi} |\langle \psi_{\omega,1}|\phi_+ e^{ip(t-x)}\rangle|^2$$

where $\langle \psi_{\omega,1}|\phi_+ e^{ip(t-x)}\rangle = \displaystyle\int_0^\infty dx e^{-ipx}(ax)^{\frac{-i\omega}{a}-\frac{1}{2}} \;\; = \frac{1}{a}(\frac{p}{a}e^{\frac{i\pi}{2}})^{\frac{i\omega}{a}-\frac{1}{2}}\Gamma(\frac{-i\omega}{a}+\frac{1}{2})$

and $L \equiv \langle \psi_{\omega,1}|\psi_{\omega,1}\rangle = \int_0^\infty \frac{du}{au} = (2\pi)\delta(0)$. We have evaluated the inner product on the hypersurface $t=0$, and we have used [37] (equation (6) page 1) in the last line. By using further properties of the Γ-function we deduce that:

$$N_{\omega,1}^{+} = \frac{1}{L}\int_0^\infty \frac{dp}{ap}\frac{1}{1+e^{\frac{2\pi\omega}{a}}} = \frac{1}{1+e^{\frac{2\pi\omega}{a}}} \tag{75}$$

as expected. Similarly,

$$N_{\omega,1}^{-} = \int_0^\infty \frac{dp}{2\pi} |\langle \psi_{-\omega,1}|\phi_+ e^{-ip(t-x)}\rangle|^2 = \frac{1}{1+e^{\frac{2\pi\omega}{a}}}$$

and it is straightforward to show that each of $N_{\omega,2}^{\pm}$ is also each equal this value.

4.3. Spatial Distribution of Rindler Particles

Although (73) and (75) each show that the spectrum of Rindler particles is thermal, they do not tell us the spatial distribution of these particles. This can be deduced from (53) and (54). For this purpose, it is clearest to work with the radar-like spatial coordinate $\chi = \frac{1}{a}\log(au)$. In these coordinates our observer is at $\chi = 0$, and $|\chi|$ represents the radar distance from the observer to the point (τ, χ) (for any τ). In these coordinates the inner product (62) takes the form $\langle\psi|\phi\rangle = \int_{-\infty}^{\infty} e^{a\chi}\psi^\dagger\phi d\chi$. The massless states of equation (74) are of the form $e^{\frac{-a\chi}{2}}\phi_\pm e^{-i\omega(\tau\mp\chi)}$. They are plane wave states in these coordinates, with the conformal factor $e^{\frac{-a\chi}{2}}$ cancelling the

$e^{a\chi}$ in the inner product. Define the particle/antiparticle densities $n^{\pm}(\tau,\chi)$ such that $n^{\pm}(\tau,\chi)d\chi$ represents the total number of particles/antiparticles within $d\chi$ of χ. From (53) and (54) this is given by:

$$n^{\pm}(\tau,\chi)=\int_0^\infty \frac{n_{\pm\omega}(\tau,\chi)}{1+e^{\frac{2\pi\omega}{a}}}\frac{\mathrm{d}\omega}{2\pi} \tag{76}$$
$$\text{where} \qquad n_\omega(\tau,\chi)=e^{a\chi I}\psi_\omega^{\dagger\mathrm{norm}}(\tau,\chi)^I\psi_\omega^{\mathrm{norm}}(\tau,\chi)$$

Both are independent of τ.

For the massless case, $\psi_\omega^\dagger\psi_\omega = e^{-a\chi}$, so that $n^+(\tau,\chi)=n^-(\tau,\chi)$ and both are spatially uniform in χ. However, as mentioned in Section 3.5, the interpretation of $n^{\pm}(\tau,\chi)$ as representing 'particle density' is only accurate when averaged over a distance L sufficient to suppress the fluctuations in $\hat{N}_L^{\pm}(\tau)$. From (55) and (56)), a straightforward calculation yields:

$$\begin{aligned}\langle(\hat{N}_L^+)^2\rangle &= \langle(\hat{N}_L^-)^2\rangle \\ &= \int_0^\infty\int_0^\infty \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\frac{\sin^2((\omega-\omega')L)}{(\omega-\omega')^2}\frac{e^{\frac{2\pi\omega'}{a}}}{1+e^{\frac{2\pi\omega'}{a}}}\frac{1}{1+e^{\frac{2\pi\omega}{a}}} \\ &= \frac{a^2L^2}{4\pi^2}\int_0^\infty \mathrm{d}\mu\frac{\sin^2(a\mu L)}{\mu^2a^2L^2}\cosh(\pi\mu)L(\mu) \qquad (77)\end{aligned}$$
$$\begin{aligned}\text{where} \qquad L(\mu) &\equiv \int_{|\mu|}^\infty \frac{d\lambda}{\cosh(\pi\mu)+\cosh(\pi\lambda)} \\ &= \frac{-\log[\cosh(\pi\mu)(\sqrt{\cosh^2(\pi\mu)+1}-1)]}{\pi\sqrt{\cosh^2(\pi\mu)+1}}\end{aligned}$$

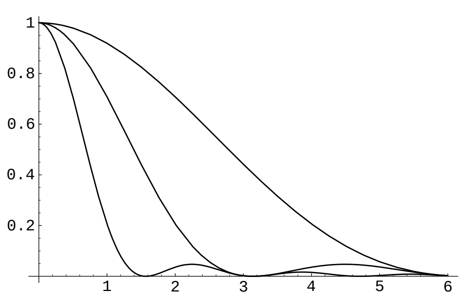

FIG. 4 A. $\frac{\sin^2(a\mu L)}{\mu^2a^2L^2}$ as a function of μ, for $L=\frac{2}{a}$ (bottom curve) $\frac{1}{a}$ and $\frac{1}{2a}$ (top curve).

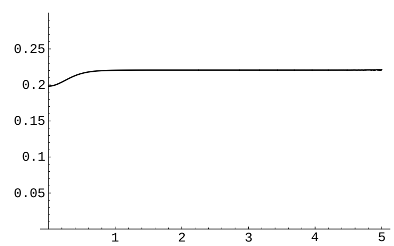

FIG 4 B. $\cosh(\pi\mu)L(\mu)$ as a function of μ. This function is $\frac{\log(\sqrt{2}+1)}{\pi\sqrt{2}}$ at $\mu=0$, but quickly approaches its large μ limit of $\frac{\log(2)}{\pi}$.

Figure 4 B shows $\cosh(\pi\mu)L(\mu)$ as a function of μ, while Figure 4 A shows $\frac{\sin^2(a\mu L)}{\mu^2 a^2 L^2}$ for $L = \frac{2}{a}$ (bottom curve) $\frac{1}{a}$ and $\frac{1}{2a}$ (top curve). For $L >> \frac{1}{a}$ the integral in (77) is dominated by contributions from small μ. We can then approximate $\cosh(\pi\mu)L(\mu) \approx \frac{\log(\sqrt{2}+1)}{\pi\sqrt{2}} \approx .198$. In this limit:

$$\langle(\hat{N}_L^+)^2\rangle \approx aL\frac{\log(\sqrt{2}+1)}{8\pi^2\sqrt{2}} \approx .45 N_L^+ \tag{78}$$

For $L >> \frac{1}{a}$ we can approximate $\cosh(\pi\mu)L(\mu) \approx \frac{\log(2)}{\pi}$, so that:

$$\langle(\hat{N}_L^+)^2\rangle \approx aL\frac{\log(2)}{8\pi^2} = .5 N_L^+ \tag{79}$$

In either case, we confirm that:

- $\langle(\frac{\hat{N}_L^+}{L})^2\rangle \approx \frac{1}{2L}\frac{N_L^+}{L} \propto \frac{1}{L}$, which would be infinite for $L \to 0$, but is finite other wise.
- The uncertainty in the number of particles in length L, given by $\sqrt{\langle(\hat{N}_L^+)^2\rangle}$, is proportional to the square root of the number of particles in that length. For this uncertainty to be less than the measurement, we must average over a length such that $N_L^+ > 1$. This is a reasonable requirement - if we want a reliably defined particle density, we must average over a volume that contains on average more than 1 particle.

Now we study the massive case. We can show from properties of the Hankel functions [37] that $n_\omega(\tau,\chi) = n_{-\omega}(\tau,\chi)$, so that the 'principle of detailed balance' [35] applies at every point in Rindler space. The relation $n_\omega(\tau,\chi) = n_{-\omega}(\tau,\chi)$ also expresses the fact that the distribution of Rindler antiparticles exactly matches the distribution of Rindler particles.

In Figure 5 we have plotted $n_\omega(\tau,\chi)/a$ as a function of $a\chi$, for $m = a$ and $\omega = \frac{a}{4}$, a and $4a$. With c and $\hbar$ included, a length $a\triangle\chi = 1$ corresponds to $\triangle\chi = \frac{c^2}{a}$. Masses are measured in units of $\frac{a\hbar}{c^3}$ and frequencies in units of $\frac{a}{c}$. Normalisation of the massless plane wave states to $(2\pi)\delta(\omega-\omega')$ represents a norm of '1 per unit length', as is conventional. For massive states this interpretation is valid only as a Cauchy principal value, so that $\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} N_\omega(\tau,\chi)d\chi \to 1$ as $L \to \infty$.)

Figure 5 shows that for negative χ the Rindler particles are uniformly distributed, but for positive χ (i.e. to the observer's right) the number of particles decreases rapidly. This fact can be understood [35] by writing equation (65) in terms of χ as:

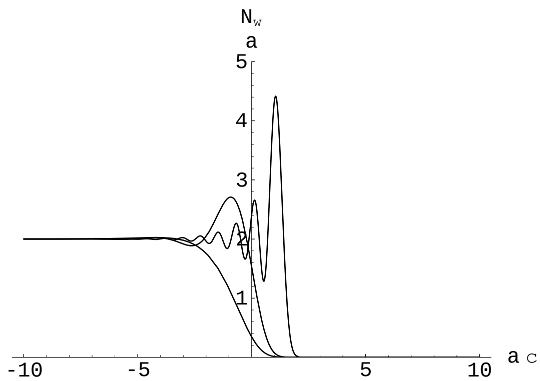

FIG. 5. $N_\omega(\tau,\chi)/a$ as a function of $a\chi$, for $m=a$, and $\omega=\frac{a}{4}$ (lowest curve), a and $4a$ (most oscillatory curve).

$$-\frac{d^2 f}{d\chi^2}+m^2 e^{2a\chi} f=(\omega+\frac{ia}{2})^2 f \tag{80}$$

which takes the form of a 1-dimensional Schrödinger equation with potential $V=m^2e^{2a\chi}$. For $m=0$ the potential disappears and we obtain the plane wave states described above. For $a=0$ the potential allows $V=m^2$ and the mass gap reappears. For nonzero m,a the potential permits states of any frequency, but does not allow them to propagate further to the right than $\chi=\frac{1}{a}\log(\frac{\omega}{m})$ before being exponentially damped. We also see from the form of (80) that a change in mass by a factor e^{al} has the effect of translating the potential a distance l to the left. This is illustrated in Figure 6.

Figure 7 (A) depicts $n(\tau,\chi)/a$ as a function of $a\chi$, for $m=a/10$ (right curve), a, and $10a$, obtained by integrating (76) numerically over $0\leq\omega\leq 2a$. The factor $\frac{1}{1+e^{\frac{2\pi\omega}{a}}}$ ensures that the number of particles present having energy $>2a$ is negligible (see Fig. 8(B)). We would like to interpret $n(\tau,\chi)/a$ as the number density at χ, and $\frac{n_\omega(\tau,\chi)}{a(1+e^{\frac{2\pi\omega}{a}})}$ as 'the number density at χ of particles of frequency ω'. This again requires caveats regarding fluctuations, in the same way as in the massless case. Accordingly, we find that the particle number density is uniform to the observer's left and negligible to the observer's right, and that the position of this transition is determined by the ratio $\frac{m}{a}$.

For $m\to 0$ this transition point goes to ∞, reproducing the spatial uniformity of the massless limit. However, for non-zero m and realistic accel-

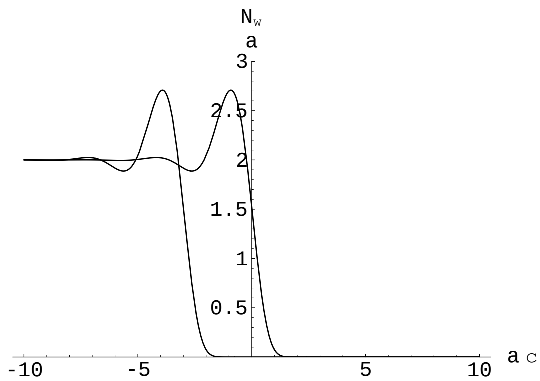

FIG. 6. $N_\omega(\tau,\chi)/a$ as a function of $a\chi$, for $w=a$, and $m=a$ (right curve)and ae^3 (left curve).

erations, the particle density at low χ (which is $\propto a$) becomes small, and the transition to a negligible density occurs far to the observer's left.

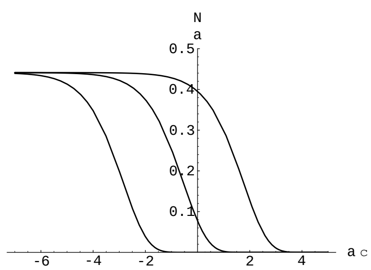

FIG. 7 A. $N(\tau,\chi)/a$ as a function of $a\chi$, for $m=a/10$ (right curve), a, and $10a$ (left curve).

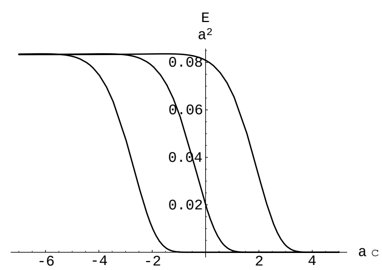

FIG 7 B. $E(\chi)/a^2$ as a function of $a\chi$, for $m=a/10$ (right curve), a, and $10a$ (left curve).

Consider now the energy density of these Rindler particles as measured by the Rindler observer. It is given by $E(\chi)=E^+(\chi)+E^-(\chi)$ where

$$E^\pm(\chi)=\int_0^\infty \frac{\omega n_{\pm\omega}(\tau,\chi)}{1+e^{\frac{2\pi\omega}{a}}}\frac{\mathrm{d}\omega}{2\pi}$$

Since $n_\omega(\tau,\chi)$ is even in ω and is independent of τ, it follows that $E^+(\chi)=E^-(\chi)$, and is independent of τ as expected. Figure 7 (B) shows $\frac{E(\chi)}{a^2}$ as a function of $a\chi$, for $m=a/10$ (right curve), a, and $10a$. This is qualitatively the same as Figure 7 (A), as expected.

It is not surprising that the Rindler observer detects the presence of particles and of energy; this is fully consistent with detector models. It

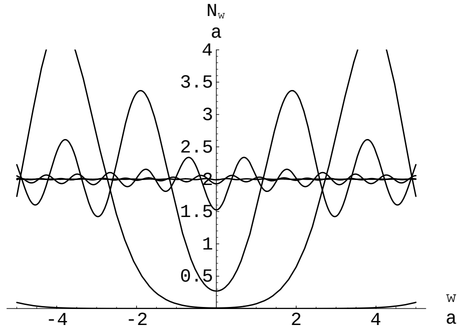

FIG. 8 A. $N_\omega(\tau,\chi)/a$ as a function of $\frac{\omega}{a}$ for $m=a$ and $a\chi=-6,-4,-2,0,1$ and 2

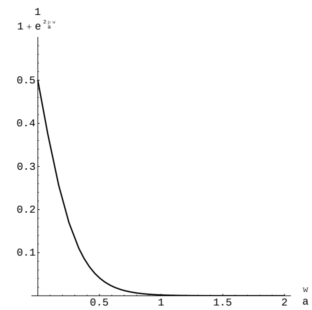

FIG 8 B. $(1+e^{\frac{2\pi\omega}{a}})^{-1}$ as a function of $\frac{\omega}{a}$.

is changes in the energy levels of the detector that signal the detection of Rindler particles. However, we should not identify the 'energy as measured by an accelerating observer' with the covariant energy momentum tensor $\langle T_{\mu\nu}\rangle$, which is zero in the Minkowski vacuum, albeit with non-zero fluctuations. To understand why there is no conflict between $E(\chi)\neq 0$ and $\langle T_{\mu\nu}\rangle = 0$, it is instructive to recall the classical connection between $T^{\mu\nu}$ and E. For instance ([38], pg 131) if we contract $T^{\mu\nu}$ with a 4-velocity u_μ, defined at at some event x, then $T^{\mu\nu}u_\nu$ represents the "4-momentum per unit of three-dimensional volume, '$\frac{dp^\mu}{dV}$', as measured in the observer's Lorentz frame at event x". This interpretation necessarily involves division by a small 3-volume dV, which must be sufficiently small that the observer's Lorentz frame remains a good description of his 'rest frame' (and also that $T^{\mu\nu}$ be effectively constant across V). This in turn requires the volume to have dimensions $\triangle x << c^2a^{-1}$ where a is the observers acceleration ([38], pg 169). But the uncertainty principle requires that $\triangle x > \frac{\hbar}{mc}$ (for the uncertainty in the momentum to be less than mc). Hence, the connection between $T^{\mu\nu}u_\nu$ and the "4-momentum as measured by an observer" applies only when $a << \frac{mc^3}{\hbar}$ - it breaks down precisely when the Unruh effect becomes significant.

We have not discussed either $\langle T_{\mu\nu}\rangle$ or the backreaction equation $G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle$ which effectively defines it. These issues are discussed extensively in the literature (see for example [7, 23, 33] and references therein). We do not add to this discussion, except to emphasise that there is no conflict between that literature and the observer-dependence discussed here. Though the Hermitian operators representing 'observables' may be non-local and observer-dependent, these do not affect the evolution of states, which is

still local and causal [1, 16]. There is no confict between seeking observer-dependent 'observables' and seeking an observer-independent semiclassical evolution equation.

Figure 8 (A) depicts $n_\omega(\tau,\chi)/a$ as a function of $\frac{\omega}{a}$ for $m=a$ and $a\chi=-6,-4,-2,0,1$ and 2. We have also shown $\omega<0$ on this plot, to demonstrate that $n_\omega(\tau,\chi)/a=n_{-\omega}(\tau,\chi)/a$ as claimed earlier. This detailed balance also implies that the distribution of Rindler antiparticles exactly matches the distribution of Rindler particles. For $\chi=\frac{-6}{a}$, where the Rindler particles are uniformly distributed, $n_\omega(\tau,\chi)/a$ is completely flat, so that the spectrum of particles is exactly thermal there. At $\chi=\frac{-4}{a}$ we see slight deviations from a thermal spectrum, becoming more pronounced as χ increases. At $\chi=\frac{1}{a}$ the deviations from thermal are such that the low frequency modes are greatly suppressed. The factor of $(1+e^{\frac{2\pi\omega}{a}})^{-1}$ in (76) (plotted in FIG. 8 (B)) suppresses all modes of frequency $\omega>a$, so that overall there are very few particles present at $\chi=\frac{1}{a}$, in accord with the middle curve of Figures 7 (A) and 7 (B).

5. FURTHER EXAMPLES OF RADAR TIME

The radar time of uniformly accelerating observers in 1+1 or 3+1 dimensions has been presented here and elsewhere [6, 16, 39, 40]. A larger family of observers in 1+1 dimensions, including the 'instant turnaround twin' (Langevin observer), and the 'gradual turn-around' twin of Figure 3 have been described in detail in [6]. In this Section we will further generalise these results. In Section 5.1 we present the radar time and radar distance of an arbitrary observer in 1+1 dimensional flat space and investigate some of their properties. Section 5.2 extends this to arbitrary observers in an arbitrary 1+1 dimensional spacetime. Sections 5.3 and 5.4 treat cosmological applications. We shall present the radar time of a comoving observer in an FRW universe of arbitrary scale factor $a(t)$ in 1+1 and in 3+1 dimensions, and examine in more detail some examples, including the Milne and deSitter universes (for the latter see also [16]).

5.1. 1+1 Dimensional Flat Space

The path of an arbitrary observer is completely characterised by a non-decreasing function $z_+^{(\gamma)}(z_-)$, where $z_\pm=t\pm z$ are null coordinates, and the proper time is given by $d\tau^2=dz_+dz_-=\frac{dz_+^{(\gamma)}}{dz_-}dz_-^2$ on the curve. Define the

functions $k_\pm(z_\pm)$ and $\tau^\pm(z_\pm)$ by:

$$k_-(z_-) \equiv \sqrt{\frac{dz_+^{(\gamma)}(z_-)}{dz_-}} \qquad k_+(z_+) \equiv \sqrt{\frac{dz_-^{(\gamma)}(z_+)}{dz_+}} = \frac{1}{k_-(z_-^{(\gamma)}(z_+))} \tag{81}$$

$$\tau^+(z_+) \equiv \int_0^{z_+} k_+(u)du \qquad \tau^-(z_-) \equiv \int_0^{z_-} k_-(u)du \tag{82}$$

The origin is chosen to coincide with the point $\tau = 0$ on the curve. On the observer's trajectory $k_- dz_- = k_+ dz_+$ and $\tau^+ = \tau^- =$ proper time. However, these functions are also defined off the trajectory. For points to the right of the observer, $\tau^\pm$ are as defined in Section 3, and for points to the left of the observer the roles of $\tau^\pm$ are reversed for later convenience. We can write the radar time and radar distance as:

$$\tau(z_+, z_-) = \tfrac{1}{2}(\tau^+(z_+) + \tau^-(z_-)) = \tfrac{1}{2}\left(\int_0^{z_+} k_+(u)du + \int_0^{z_-} k_-(u)du\right)$$
$$\rho(z_+, z_-) = \tfrac{1}{2}|\tau^+(z_+) - \tau^-(z_-)| = \tfrac{1}{2}\left|\int_0^{z_+} k_+(u)du - \int_0^{z_-} k_-(u)du\right|$$

To make this construction more concrete, consider the case of an inertial observer. In this case $z_+^{(\gamma)}(z_-) = k_-^2 z_-$ where $k_\pm = \sqrt{\frac{1\mp v}{1\pm v}} =$ constant. k_- is the "k" of Bondi's 'k-calculus'. $\tau^\pm = k_\pm z_\pm$, so that the radar time is

$$\tau = \tfrac{1}{2}\sqrt{\frac{1-v}{1+v}}z_+ + \tfrac{1}{2}\sqrt{\frac{1+v}{1-v}}z_- \qquad = \frac{t - vx}{\sqrt{1-v^2}}$$

which is the time coordinate of the observer's rest frame, as expected.

For a uniformly accelerating observer we have $z_+^{(\gamma)}(z_-) = \frac{z_-}{1-az_-}$, where the curve has been translated relative to Section 5 so that it passes through the origin. This gives $k_\pm(z_\pm) = \frac{1}{1\pm az_\pm}$, so that:

$$\tau = \frac{1}{2a}\log(\frac{1+az_+}{1-az_-}) = \frac{1}{2a}\log(\frac{x+\frac{1}{a}+t}{x+\frac{1}{a}-t})$$
$$\rho = \frac{1}{2a}|\log((1+az_+)(1-az_-))|$$

which are the translated versions of (59), as expected.

Returning to the general case, we can write the metric in coordinates (τ,ρ) as:

$$ds^2 = \frac{d\tau^2 - d\rho^2}{k_+(z_+)k_-(z_-)}$$

To write$k_\pm(z_\pm)$ in terms of $\tau_\pm$ involves inverting the expressions (82) for $\tau_\pm(z_\pm)$, which is always possible since $\tau_\pm(z_\pm)$ are both strictly increasing functions (for a future-directed timelike observer). It is easy to verify that $k_+k_- = 1$ for the inertial observer, and that $k_+k_- = e^{-2a\rho}$ for the uniformly accelerating observer. The time-translation vector field, defined in (16), is $\frac{\partial}{\partial\tau}$ for all observers.

Finally, note that $\nabla^2\tau = 0 = \nabla^2\rho$, and $\frac{\partial\tau}{\partial z} = \frac{\partial\rho}{\partial t}$ and $\frac{\partial\tau}{\partial t} = \frac{\partial\rho}{\partial z}$. Hence, if we Wick rotate $(t,z) \to (s \equiv it, z)$ and $(\tau,\rho) \to (\xi \equiv i\tau, \rho)$ it follows that ξ, ρ are harmonic, and $\xi + i\rho$ is an analytic (conformal) function of $s + iz$.

5.2. Arbitrary 1+1D Spacetime

All 1+1 dimensional spacetimes can be written in the form:

$$ds^2 = \Omega^2(z_+,z_-)dz_+dz_- = \Omega^2(dt^2 - dz^2)$$

where $z_\pm = t \pm z$ as before. Much of the previous subsection carries over, with the main difference being the parametrisation of proper time, which must now satisfy $d\tau^2 = \Omega^2 dz_+dz_-$ on the curve.

Define:

$$\begin{aligned}
k_-(z_-) &\equiv \Omega(z_+^{(\gamma)}(z_-), z_-)\sqrt{\frac{dz_+^{(\gamma)}(z_-)}{dz_-}} \\
k_+(z_+) &\equiv \Omega(z_+, z_-^{(\gamma)}(z_+))\sqrt{\frac{dz_-^{(\gamma)}(z_+)}{dz_+}} = \frac{\Omega(z_+, z_-^{(\gamma)}(z_+))^2}{k_-(z_-^{(\gamma)}(z_+))} \\
\tau^+(z_+) &\equiv \int_{z_0}^{z_+} k_+(u)du \qquad \tau^-(z_-) \equiv \int_{z_-^{(\gamma)}(z_0)}^{z_-} k_-(u)du
\end{aligned}$$

We have still chosen the origin of the (z_+,z_-) coordinate system to lie on the observers trajectory, but we no longer require that it coincide with the point of zero proper time. The point $\tau = 0$ now occurs at $(z_+,z_-) = (z_0, z_-^{(\gamma)}(z_0))$. This is for later convenience. We still have $\tau(z_+,z_-) = \frac{1}{2}(\tau^+(z_+) + \tau^-(z_-))$ and $\rho(z_+,z_-) = \frac{1}{2}|\tau^+(z_+) + \tau^-(z_-)|$, and the metric becomes:

$$ds^2 = \frac{\Omega^2(z_+,z_-)}{k_+(z_+)k_-(z_-)}(d\tau^2 - d\rho^2) \tag{83}$$

The time-translation vector field is still $\frac{\partial}{\partial\tau}$, and the final comments of Section 5.1, about the analyticity of the Wick-rotated spacetime, still apply.

5.3. FRW Universes in 1+1 Dimensions

Consider a comoving observer in an FRW universe of arbitrary scale factor $a(t)$, in 1+1 dimensions:

$$ds^2 = dt^2 - a(t)^2 dz^2 = C(\eta)^2(d\eta^2 - dz^2)$$

where $C(\eta(t)) = a(t)$ and:

$$\eta(t) = \eta_0 + \int_0^t \frac{dt'}{a(t')} \tag{84}$$

This is a special case of Section 5.2. A comoving observer at $z = 0$ satisfies $z_+^{(\gamma)}(z_-) = z_-$ where $z_\pm \equiv \eta \pm z$, so that

$$k_\pm(z_\pm) = C(z_\pm) \text{ and } \tau^\pm(z_\pm) = \int_{\eta_0}^{z_\pm} C(u)du \tag{85}$$

We have chosen $z_0 = \eta_0$, so that on the observer's trajectory $\tau^+ = \tau^- = t$ (which is the observer's proper time). From (83), the metric can be written as

$$ds^2 = \frac{C(\frac{1}{2}(z_+ + z_-))^2}{C(z_+)C(z_-)}(d\tau^2 - d\rho^2) \tag{86}$$

In this case we can invert $\tau^\pm(z_\pm)$ explicitly, to get

$$z_\pm = \eta(\tau^\pm) \tag{87}$$

(with $\eta(u)$ defined as in (84)). This allows us to rewrite (86) in terms of τ, ρ as:

$$ds^2 = \frac{C(\frac{1}{2}\eta(\tau^+) + \frac{1}{2}\eta(\tau_-))^2}{a(\tau^+)a(\tau^-)}(d\tau^2 - d\rho^2) \tag{88}$$

where $\tau^\pm = \tau \pm \rho$.

To generate insight into these results, we consider 3 examples.

5.3.1. *deSitter Space*

DeSitter space has $a(t) = e^{\lambda t}$, which gives $\eta(t) = \frac{-1}{\lambda}e^{-\lambda t}$ and $C(\eta) = \frac{-1}{\lambda\eta}$. Note that $\eta(t) < 0$ for all t, so the domain of $\tau(x)$ will cover only the causal

past of the point $(\eta,z)=(0,0)$, with particle horizons at $z=\mp\eta=\frac{\pm 1}{\lambda}e^{-\lambda t}$. From (85) or (87) we have $\tau^{\pm}(z_{\pm})=\frac{-1}{\lambda}\log(-\lambda z_{\pm})$, so that τ,ρ are given by:

$$\tau=\frac{-1}{2\lambda}\log(\lambda^2 z_+ z_-)=\frac{-1}{2\lambda}\log(\lambda^2(\eta^2-z^2))$$
$$=\frac{-1}{2\lambda}\log(e^{-2\lambda t}-\lambda^2 z^2) \tag{89}$$
$$\rho=\left|\frac{1}{2\lambda}\log\left(\frac{e^{-\lambda t}+\lambda z}{e^{-\lambda t}-\lambda z}\right)\right|=\frac{1}{2\lambda}\log\left(\frac{e^{-\lambda t}+\lambda|z|}{e^{-\lambda t}-\lambda|z|}\right) \tag{90}$$

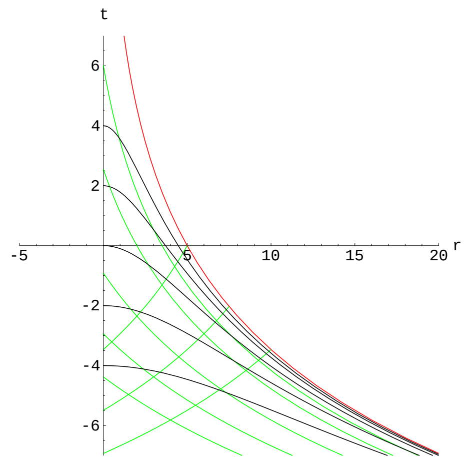

FIG. 9. DeSitter space in $(t,|z|)$ coordinates. The grey lines represent ingoing and outgoing photon trajectories (the outermost one is the particle horizon $z(t)=\frac{\pm e^{-\lambda t}}{\lambda}$), and the black lines are the observer's hypersurfaces of simultaneity for various values of τ_0.

The metric in (τ,ρ) coordinates, calculated either from (88) or directly from (89) and (90), is $ds^2=\cosh^{-2}(\lambda\rho)(d\tau^2-d\rho^2)$. The time-translation vector field is given by:

$$k=\frac{\partial}{\partial t}-\lambda z\frac{\partial}{\partial z} \qquad \text{or } =\frac{\partial}{\partial \tau} \text{ in } (\tau,\rho) \text{ coordinates}$$

Again this is a timelike Killing vector field on the domain of $\tau(x)$, and is spacelike outside this region. This τ coordinate, and the corresponding Killing vector field, are the same as those used in Gibbons and Hawkings original derivation of the thermal deSitter spectrum in 1977 [41], and as

in various other derivations since [42, 43] (see also [7] and the references therein). The metric takes a more familiar form [44] if we substitute $u = \frac{\tanh(\lambda\rho)}{\lambda}$, giving $ds^2 = (1-\lambda^2u^2)d\tau^2 - \frac{du^2}{1-\lambda^2u^2}$. Hypersurfaces of constant τ have been plotted in Figure 9.

5.3.2. The Milne Universe

The 1+1 D Milne universe is given by $ds^2 = dt^2 - a_0^2t^2dz^2$. Application of (85) now gives $\tau^{\pm} = te^{\pm a_0 z}$, so that:

$$\tau(x) = t\cosh(a_0 z) \qquad \rho(x) = |t\sinh(a_0 z)|$$

In terms of which the metric is $ds^2 = d\tau^2 - d\rho^2$. It is convenient to drop the absolute value in ρ and to treat ρ as a spatial (rather than a radial) coordinate. Although the cases $t > 0$ and $t < 0$ must be treated separately, a similar transformation holds in each case. With the coordinate singularity at $t = 0$ moved to the light cone through the origin, the regions $t > 0$ and $t < 0$ are revealed to be regions F and P of Rindler space, and τ, ρ are the inertial coordinates of the underlying flat space. This is illustrated in Figure 10.

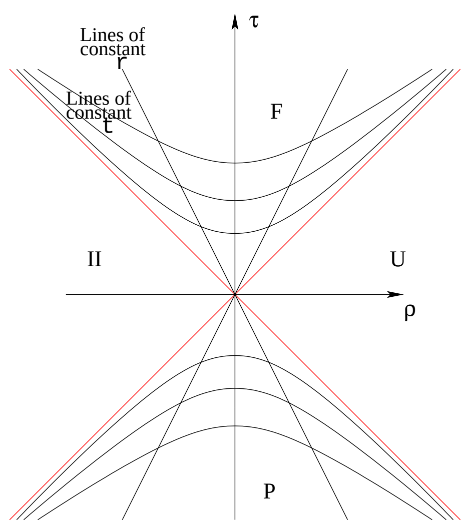

FIG. 10. The Milne universe in (τ,ρ) coordinates. The (t,r) coordinate system corresponds to regions F and P of Section 5, but do not correspond to the radar coordinates of any observer. The radar coordinates (τ,ρ) of the 'comoving' (inertial) observer recover the underlying flat space.

Since $t=0$ is just a coordinate singularity, there is no reason for the observer's trajectory to terminate there; the 'completed' geodesic will cover all t. Clearly, as this observer charts her radar coordinates (sending and receiving light from various spacetime points) it will be readily apparent to her that regions II and U also exist, and the domain of (τ,ρ) will certainly include these regions. The lines $\tau=\pm\rho$ which bound the original (t,z) coordinate system are of no significance in the (τ,ρ) coordinate system.

Some approaches to particle creation in the Milne universe (see [7] Section 5.3, and references therein) exploit the similarity with the Rindler case to derive non-trivial quantum effects in this spacetime. However, an important distinction must be made. Region U of Rindler space, expressed in Rindler coordinates, corresponds to spacetime as seen by a uniformly accelerating observer in flat space; it is this fact that gives physical meaning to the Rindler coordinate system. In the Milne universe, however, *no* observer would have t as their radar time. A comoving observer at the origin of the Milne universe corresponds simply to an inertial observer (stationary at the origin) in flat space. It is therefore reassuring that the radar time of that observer brings us back to Minkowski space, and to the standard inertial vacuum.

5.3.3. Radiation domination

In this case $a(t)=\sqrt{\lambda t}$, which gives:

$$\tau(x)=t+\frac{\lambda}{4}z^2 \qquad \rho(x)=|z|\sqrt{\lambda t}$$
$$ds^2=\tfrac{1}{2}\left(1+\frac{\tau}{\sqrt{\tau^2-\rho^2}}\right)(d\tau^2-d\rho^2)$$

Hypersurfaces of constant radar time are plotted in Figure 11.

As a result of the singularity at $t=0$, the comoving observer's geodesic starts at finite proper time (chosen for convenience to be zero). The horizon now represents the future light cone $\rho=\tau$ (or $r=2\sqrt{\frac{t}{\lambda}}$) of the origin. This is not an acceleration horizon, and (unlike the Rindler case) the time translation vector field does not tend to zero there. Instead we find that, though the direction of the vector field $k^\mu(x)$ becomes null on the horizon, its components become infinite in this limit, so that $k^2\to\infty$ as the horizon is approached (i.e. as $\rho\to\tau$).

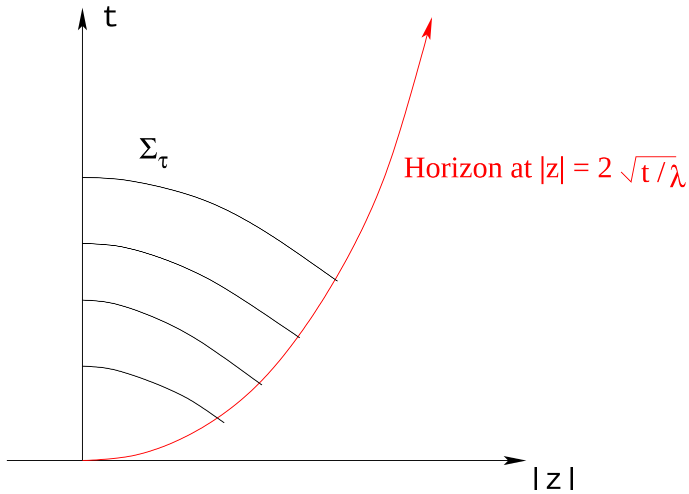

FIG. 11. Hypersurfaces of constant τ in an FRW universe with scale factor $a(t)=\sqrt{\lambda t}$, in $(t,|z|)$ coordinates. The domain of radar time covers only the causal future of the origin.

5.4. 3+1 D Cosmologies

Consider the metric

$$ds^2 = dt^2 - a(t)^2\left(dr^2 + f(r)^2(d\theta^2 - \sin^2(\theta)d\phi^2)\right)$$

(where $f(r)=\sin(r), r$ or sinh(r) for spatial sections which are respectively hyperbolic, flat, or closed), and consider a comoving observer at the origin. Only the radial null geodesics are relevant for determining $\tau^\pm$ for this observer, so that we can define $z^\pm = \eta \pm r$ and the 1+1 D results carry over almost unaltered. The $\tau^\pm$ are still given by (85) or by inverting (87), and the metric is given by:

$$ds^2 = C(\tfrac{1}{2}\eta(\tau^+) + \tfrac{1}{2}\eta(\tau^-))^2\left(\frac{d\tau^2 - d\rho^2}{a(\tau^+)a(\tau^-)} + f(r(\tau^+,\tau^-))^2(d\theta^2 - \sin^2(\theta)d\phi^2)\right)$$

where $r(\tau^+,\tau^-) = \frac{1}{2}(\eta(\tau^+) - \eta(\tau^-))$. The examples of Section 5.3 extend almost unaltered to the 3+1 dimensional case.

6. DISCUSSION

We have presented a formulation of fermionic quantum field theory in electromagnetic and gravitational backgrounds that is analogous to the

methods used in multiparticle quantum mechanics. The main difference is in the particle interpretation, which requires us to consider the entire Dirac Sea, as well as any particles which may be present. This approach provides a conceptually transparent approach to the theory and a simple derivation of the general S-Matrix element and expectation value of the theory. Moreover, it also leads to a consistent particle interpretation for all times and any background, without requiring any 'asymptotic niceness conditions' on the 'in' and 'out' states. Other advantages include the ease with which unitarity of the S-Matrix follows from conservation of the Dirac inner product, insights into quantum anomalies, and the fact that Hermitian extension provides well-defined second quantized operators without requiring a complete set of orthonormal modes. We have used the concept of 'radar time' to generalise the particle interpretation to an arbitrarily moving observer, providing a definition of particle which depends only on the observer's motion and on the background, and not on the choice of coordinates, the choice of gauge, or the detailed construction of the particle detector.

Ever since the pioneering work of Unruh [31] and Davies [32] in 1975, it has been known that the concept of particle differs for different observers. However, attempts to systematically assign a choice of particle/antiparticle modes uniquely to a given observer have not been successful. Such definitions have either relied on the existence of suitable symmetries (Killing vectors, conformal symmetry etc) or on an arbitrary choice of foliation of spacetime into 'space' and 'time' (see for instance [7, 33] and references therein). A notable exception is reference [45] which treated arbitrary observers in flat space, using a foliation defined uniquely in terms of the motion of the observer. However, as well as being only applicable in flat space, their foliation is often multivalued, and omits portions of the observers causal envelope if discrete changes in velocity are allowed [14].

Though particle detector models provide a useful operational particle concept, it is inherently circular for a particle detector to be anything that detects particles, and a particle to be 'anything detected by a particle detector'. It is also known [4, 5] that in electromagnetic backgrounds (and in cases where particles could already be defined independtly of detector models) the predictions of particle detector models are not always proportional to the number of particles present, even when the detector is inertial. We might conclude that a particle concept is only an approximate notion, or even that 'particles do not exist' [46]. However, the observer-dependent particle interpretation presented here (and in [1]) averts this pessimistic

conclusion, and provides a concrete answer to to the question "what do particle detectors detect?"

We hope that the present work, along with [1, 16], has shown the computational and conceptual value of working with a concrete representation of the Dirac Sea. We strongly support Jackiw's claim [17] that "physical consequences can be drawn from Dirac's construction". Care has been taken to ensure that the dynamics is kept separate from the kinematics. The evolution equation is explicitly local and causal. Though the categorisation of states in terms of their particle content requires an observer-dependent foliation of spacetime, this in no way affects the evolution of these states. This goes some way towards showing that the foliation dependence of quantum mechanics need not conflict with the coordinate covariance of general relativity, provided one remembers the important role played in both theories by the observer.

APPENDIX: GREEN FUNCTIONS AND PROJECTION OPERATORS

In equations (39) - (40) the vacuum expectation value of a general 'one-particle' operator $\hat{A}_{\text{phys}}$ (the physical extension of some operator $\hat{A}_1 : \mathcal{H} \to \mathcal{H}$) was expressed in terms of traces of projection operators $\hat{P}^{\pm}_{|F\rangle}(\tau)$ and $\hat{P}^{\pm}_{\tau}$. It is common in many textbooks [7, 33, 8] to express expectation values in terms of traces of (2-point) Greens functions. This appendix summarises the connections between projection operators and Greens functions. We also make connection with the "first order density matrix" or "Dirac density matrix" of multiparticle quantum mechanics (see [30] pgs 8-10 for instance), emphasising the role of the negative-energy Wightman function as the Dirac density matrix of the Dirac Sea. The observer-dependent particle interpretation presented in Section 3, along with the corresponding time-dependent vacuum, allows the definition of a time-dependent family of 'vacuum Greens functions'. By presenting these as special cases of general state-dependent Greens functions, further clarification can be made of the connection between the negative-energy Wightman function and the Dirac density matrix.

In the quantum mechanics of non-relativistic fermions, a common tool (see e.g. [30], pg 8-10) is the *first order density matrix* $\gamma(\boldsymbol{r}',\sigma',\boldsymbol{r},\sigma)$, defined from the normalised many-body wave function $\Phi(\boldsymbol{r}_1,\sigma_1,\ldots,\boldsymbol{r}_N,\sigma_N)$ by:

$$\gamma_\Phi(\boldsymbol{r}',\sigma',\boldsymbol{r},\sigma)=N\sum_{\sigma_2,\ldots,\sigma_N}\int \mathrm{d}\boldsymbol{r}_2\ldots\mathrm{d}\boldsymbol{r}_N\,\Phi(\boldsymbol{r},\sigma,\ldots,\boldsymbol{r}_N,\sigma_N)\Phi^*(\boldsymbol{r}',\sigma',\ldots,\boldsymbol{r}_N,\sigma_N)$$

When the wavefunction is a Slater determinant of one-particle states ϕ_i, this takes the simpler form:

$$\gamma_\Phi(\boldsymbol{r}',\sigma',\boldsymbol{r},\sigma)=\sum_{i=1}^{N}\phi_i(\boldsymbol{r},\sigma)\phi_i^*(\boldsymbol{r}',\sigma')$$

and is referred to as the *Dirac density matrix*. Consider now the relativistic case, and take the multiparticle wavefunction to be the 'evolved vacuum' $|\mathrm{vac}_{\tau_0}(\tau)\rangle$ from (22). Then the Dirac density matrix 'at time τ' is:

$$\gamma_{|\mathrm{vac}_{\tau_0}(\tau)\rangle}(y|_\tau,x|_\tau)=\sum_i v_{i,\tau_0}(x|_\tau)\bar{v}_{i,\tau_0}(y|_\tau)$$

(which is a 4×4 matrix, implicitly containing the spin dependence). From (37) this is simply the negative energy projection operator $\hat{P}^-_{\tau_0}(\tau)$ in coordinate representation. More precisely:

$$\hat{P}^-_{\tau_0}(\tau)\psi(x|_\tau)=\int_{\Sigma_\tau} e(y)\gamma_{|\mathrm{vac}_{\tau_0}(\tau)\rangle}(y|_\tau,x|_\tau)\gamma^\mu(y)\psi(y|_\tau)d\Sigma_\mu(y)$$

The connection between $\hat{P}^-_{\tau_0}(\tau)$, $\gamma_{|\mathrm{vac}_{\tau_0}(\tau)\rangle}(y,x)$ and the *negative energy Wightman function* $S^-(x,y)\equiv\langle 0|\bar{\hat{\psi}}(y)\hat{\psi}(x)|0\rangle$ also follows. To see this, introduce the Schrödinger picture field operator $\hat{\psi}(x|_\Sigma)\equiv\sum_i\psi_i(x|_\Sigma)i_{\psi_i(x|_\Sigma)}$, as in multiparticle quantum mechanics. The connection with Canonical methods is straightforward [1]. The field operator can be written in the Heisenberg picture as $\hat{\psi}(x)=\sum_i\psi_i(x)i_{\psi_i(x|_{\Sigma_r})}$, where Σ_r is an arbitrary fixed 'reference hypersurface' on which the Heisenberg picture states are defined. If we choose the 'vacuum' $|0\rangle$ to be $|\mathrm{vac}_{\tau_0}(\Sigma_\mathrm{r})\rangle$ (which is the Heisenberg picture representative of $|\mathrm{vac}_{\tau_0}(\tau)\rangle$) then we have

$$S^-_{\tau_0}(x,y)=\sum_i v_{i,\tau_0}(x)\bar{v}_{i,\tau_0}(y)$$

where the subscript τ_0 signifies that $S^-_{\tau_0}(x,y)$ depends on τ_0 through the choice of vacuum state $|\mathrm{vac}_{\tau_0}(\tau)\rangle$. Therefore $\gamma_{|\mathrm{vac}_{\tau_0}(\tau)\rangle}(y|_\tau,x|_\tau)=S^-_{\tau_0}(x|_\tau,y_\tau)$. That is, the negative energy Wightman function is the Dirac density function of the Dirac Sea, and is the kernel of $\hat{P}^-_{\tau_0}(\tau)$. The *positive energy*

Wightman function $S^+_{\tau_0}(x,y)$ can similarly be written as:

$$S^+_{\tau_0}(x,y) \equiv \rangle \mathrm{vac}_{\tau_0}(\Sigma_\mathrm{r})|\hat{\psi}(\mathrm{x})\bar{\hat{\psi}}(\mathrm{y})|\mathrm{vac}_{\tau_0}(\Sigma_\mathrm{r})\rangle = \sum_\mathrm{i} \mathrm{u}_{\mathrm{i},\tau_0}(\mathrm{x})\bar{\mathrm{u}}_{\mathrm{i},\tau_0}(\mathrm{y})$$

which is the kernel of $\hat{P}^+_{\tau_0}(\tau)$.

In terms of $S^\pm_{\tau_0}(x,y)$, other commonly used 2-point functions [33] are the:

Feynman propogator	$iS_{F,\tau_0}(x,y)$	$=\theta(\tau_x-\tau_y)S^+_{\tau_0}(x,y)-\theta(\tau_y-\tau_x)S^-_{\tau_0}(x,y)$
Hadamard function	$S^{(1)}_{\tau_0}(x,y)$	$=S^+_{\tau_0}(x,y)-S^-_{\tau_0}(x,y)$
full propogator	$iS(x,y)$	$=S^+_{\tau_0}(x,y)+S^-_{\tau_0}(x,y)$
retarded propogator	$S_R(x,y)$	$=-\theta(\tau_x-\tau_y)S(x,y)$
advanced propogator	$S_A(x,y)$	$=\theta(\tau_y-\tau_x)S(x,y)$

$S^{(1)}_{\tau_0}(x|_\tau,y|_\tau)$ is the kernel of the 'first quantized' number operator $\hat{N}_{1,\tau_0}(\tau)$, while $iS(x|_\tau,y|_{\tau'})$ is the kernel of the 'first quantized' evolution operator $\hat{U}_1(\tau,\tau')$. $S(x,y)$, $S_R(x,y)$ and $S_A(x,y)$ are independent of τ_0, since they do not depend on any choice of state. Furthermore they do not depend on the decomposition of solutions into positive/negative frequency modes.

The Wightman function $S^-_{\tau_0}(x,y)$ can be generalised to an arbitrary state $|F(\Sigma_r)\rangle$ (which is the Heisenberg picture representative of $|F(\tau)\rangle$) as:

$$\begin{aligned} S^-_{|F\rangle}(x,y) &\equiv \langle F(\Sigma_r)|\bar{\hat{\psi}}(y)\hat{\psi}(x)|F(\Sigma_r)\rangle \\ &= \sum_{i\in I}\psi_i(x)\bar{\psi}(y) = \gamma_{|F\rangle}(y,x) \end{aligned}$$

Again, the negative energy Wightman function is just the Dirac density matrix of the state in question, and is the kernel of the operator $\hat{P}^-_{|F\rangle}(\tau)$ defined in Section 3.3. Here $S^+_{|F\rangle}(x,y)$ is the kernel of $\hat{P}^+_{|F\rangle}(\tau)$.

ACKNOWLEDGMENTS

We thank Anton Garrett for helpful discussions. Carl Dolby also thanks Merton College, Oxford for financial support.

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